EXTREMAL PROPERTIES OF HERMITIAN MATRICES. II

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1. Introduction. Let H be an n-square Hermitian matrix with eigenvalues $h_1 \geqslant h_2 \geqslant \ldots \geqslant h_n$. Fan **(2)** showed that

$$(1) \qquad \begin{cases} \max \sum_{j=1}^{k} (Hx_j, x_j) = \sum_{j=1}^{k} h_j, \\ \min \sum_{j=1}^{k} (Hx_j, x_j) = \sum_{j=1}^{k} h_{n-k+j} \end{cases}$$

$k = 1, 2, \ldots, n$, where the max and min are taken over all sets of k orthonormal (o.n.) vectors in unitary n-space V_n. Marcus and McGregor **(3)** have generalized this result in the case that H is non-negative Hermitian. For vectors $x_1, \ldots, x_r$, $r \leqslant n$, in V_n, let $x_1 \wedge x_2 \wedge \ldots \wedge x_r$ denote the Grassmann exterior product of the x_i; it is a vector in V_m, where

$$m = \binom{n}{r}.$$

The rth compound of H is a Hermitian transformation of V_m defined by

$$C_r(H)\, x_1 \wedge \ldots \wedge x_r = Hx_1 \wedge \ldots \wedge Hx_r.$$

For $1 \leqslant r \leqslant k \leqslant n$, denote by Q_{kr} the set of $\binom{k}{r}$ distinct sequences $w = \{i_1, \ldots, i_r\}$ of integers such that $1 \leqslant i_1 < \ldots < i_r \leqslant k$. For a set of vectors $x_1, \ldots, x_k$ in V_n, set

$$x_w = x_{i_1} \wedge \ldots \wedge x_{i_r}.$$

Let

$$(2) \qquad g = g(x_1, \ldots, x_k) = \sum_{w \in Q_{kr}} (C_r(H)x_w, x_w),$$

and let $E_r(a_1, \ldots, a_k)$ be the rth elementary symmetric function of the numbers $a_1, \ldots, a_k$. Marcus and McGregor showed that

$$(3) \qquad \begin{cases} \max g = E_r(h_1, \ldots, h_k) \\ \min g = E_r(h_{n-k+1}, \ldots, h_n), \end{cases}$$

where the max and min are taken over all sets of k o.n. vectors $x_1, \ldots, x_k$ in V_n. This result reduces to (1) when $r = 1$. In the present note we extend this result to the case where H is an arbitrary Hermitian matrix.

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2. Results.

THEOREM. *Let* $1 \leqslant r \leqslant k \leqslant n$ *and let* H *be a Hermitian matrix with eigenvalues* $h_1 \geqslant \ldots \geqslant h_n$. *Then*

$$(4)\qquad \begin{cases} \max g = \max\limits_{0 \leqslant s \leqslant k} E_r(h_1, \ldots, h_s, h_{n-k+s+1}, \ldots, h_n)^* \\ \min g = \min\limits_{0 \leqslant s \leqslant k} E_r(h_1, \ldots, h_s, h_{n-k+s+1}, \ldots, h_n), \end{cases}$$

where the max and min of g *are taken over all sets of* k o.n. *vectors* $x_1, \ldots, x_k$ *in* V_n.

Proof. Let $L = L(x_1, \ldots, x_k)$ denote the subspace spanned by the o.n. vectors $x_1, \ldots, x_k$; and let P be the orthogonal projection of V_n into L. Then, since P is Hermitian,

$$\begin{aligned} g(x_1, \ldots, x_k) &= \sum_{w \in Q_{kr}} (C_r(H)x_w, C_r(P)x_w) \\ &= \sum_{w \in Q_{kr}} (C_r(PH)x_w, x_w) \\ &= \text{trace of } C_r(A) \\ &= E_r(\lambda_1, \ldots, \lambda_k), \end{aligned}$$

where A is the Hermitian transformation PH restricted to L, and $\lambda_1 \geqslant \ldots \geqslant \lambda_k$ are the eigenvalues of A. It is known (1, p. 33) that for $1 \leqslant j \leqslant k$,

$$(5)\qquad h_j \geqslant \lambda_j \geqslant h_{n-k+j}.$$

Let $R_k(h)$ be the set of real k-tuples $\lambda = (\lambda_1, \ldots, \lambda_k)$, $\lambda_1 \geqslant \ldots \geqslant \lambda_k$, satisfying the inequalities (5). Thus the values of g are bounded by the extreme values of $E_r(\lambda) = E_r(\lambda_1, \ldots, \lambda_k)$ as λ ranges over $R_k(h)$. We shall discuss the maximum value of $E_r(\lambda)$ in the following lemmas. Corresponding results hold for the minimum. For the moment we restrict ourselves to the case in which the h_j are distinct.

LEMMA 1. *Let* $h_1 > \ldots > h_n$ *be given real numbers. Let* $1 \leqslant r \leqslant k \leqslant n$, *and let*

$$(6)\qquad \gamma = \max_{\lambda \in R_k(h)} E_r(\lambda).$$

Then there exists $\mu \in R_k(h)$ *such that*

$$(7)\qquad E_r(\mu) = \gamma$$

and $\mu_1 > \ldots > \mu_k$.

Proof. When $r = 1$, the unique solution of (7) is: $\mu_j = h_j$, $j = 1, \ldots, k$. Hence suppose that $2 \leqslant r \leqslant k$.

Let $T_{kj}(h)$ be the set of $\lambda = (\lambda_1, \ldots, \lambda_k) \in R_k(h)$ such that $E_r(\lambda) = \gamma$ and $\lambda_1 > \ldots > \lambda_j$. Then $T_{k1}(h)$ is not void by the continuity of the elemen-

*If $s = 0$ (or k) the initial (or terminal) segment is missing.

tary symmetric functions. Let m be the least integer such that $T_{km}(h)$ is not void. Then m must equal k for, if not, we shall show that there exists $\nu \in T_{k,m+1}(h)$. Suppose then that $\mu \in T_{km}(h)$, where

(8) $$\mu_1 > \ldots > \mu_m = \ldots = \mu_t > \mu_{t+1} \geqslant \ldots \geqslant \mu_k.$$

From (5) and (8) we have

(9) $$h_m > h_{m+1} \geqslant \mu_{m+1} = \mu_m = \mu_{t-1} = \mu_t \geqslant h_{n-k+t-1} > h_{n-k+t}.$$

Furthermore,

(10) $$\begin{aligned} E_r(\mu) &= \mu_m E_{r-1}(\tilde{\mu}_m) + E_r(\tilde{\mu}_m) \\ &= \mu_t E_{r-1}(\tilde{\mu}_t) + E_r(\tilde{\mu}_t) \end{aligned}$$

where $E_q(\tilde{\mu}_j)$ means $E_q(\mu_1, \ldots, \mu_{j-1}, \mu_{j+1}, \ldots, \mu_k)$. (If $r = k$, $E_r(\tilde{\mu}_j) = 0$.) Now $E_{r-1}(\tilde{\mu}_m) = E_{r-1}(\tilde{\mu}_t) = 0$. For, if $E_r(\tilde{\mu}_m) > 0$, then for $\mu' = (\mu_1, \ldots, \mu_m + \delta, \ldots, \mu_k)$,

$$E_r(\mu') = (\mu_m + \delta)\, E_{r-1}(\tilde{\mu}_m) + E_r(\tilde{\mu}_m) > E_r(\tilde{\mu})$$

for $\delta > 0$, and, by (8) and (9), $\mu' \in R_k(h)$ for δ sufficiently small. This contradicts (6). Similarly, if $E_{r-1}(\tilde{\mu}_t) < 0$, $E_r(\mu'') > E_r(\mu)$ for $\mu'' = (\mu_1, \ldots, \mu_t - \delta, \ldots, \mu_k)$. Hence $E_r(\mu) = E_r(\tilde{\mu}_m)$ is independent of μ_m. Set $\nu_j = \mu_j$ for $j \neq m$, and choose $\nu_m > \mu_m$ so that $\nu_m < h_m$ and $\nu_m < \nu_{m-1}$ (if $m > 1$). Then $\nu \in T_{k,m+1}(h)$.

LEMMA 2. *Under the hypotheses of Lemma* 1,

(11) $$\gamma = \max_{0 \leqslant s \leqslant k} E_r(h_1, \ldots, h_s, h_{n-k+s+1}, \ldots, h_n).$$

Proof. Since the lemma is obviously true when $r = 1$, and also when $k = n$, suppose that $2 \leqslant r \leqslant k < n$. By Lemma 1, $T_{kk}(h)$ is not empty. Let $S_{kq}(h)$, $1 \leqslant q \leqslant k$, be the set of those $\lambda \in T_{kk}(h)$ for which $\lambda_j = h_j$, $j = 1, \ldots, q$; and let $S_{k0}(h)$ be the set of $\lambda \in T_{kk}(h)$ for which $\lambda_1 < h_1$. Let s be the largest integer such that $S_{ks}(h)$ is not empty. If $s = k$, there is nothing to prove. Otherwise let $\mu \in S_{ks}(h)$. Then

$$\mu_j = h_{n-k+j},\; j = s + 1, \ldots, k;$$

for, if not, we shall show that there exists $\nu \in S_{k,s+1}(h)$, contradicting the choice of s.

Let t be the least integer greater than s for which $\mu_t > h_{n-k+t}$. If $t = s + 1$, $h_t > \mu_t$ by the maximality of s; while if $t > s + 1$

$$h_t \geqslant h_{n-k+t-1} = \mu_{t-1} > \mu_t.$$

Thus

$$h_t > \mu_t > h_{n-k+t}.$$

It follows that $E_{r-1}(\tilde{\mu}_t) = 0$, since otherwise we could vary μ_t up or down to increase $E_r(\mu)$ (see (10)) while keeping μ in $T_{kk}(h)$.

Thus

(12) $$E_r(\mu) = E_r(\tilde{\mu}_t).$$

Set

$$\nu_j = \mu_j, j = 1, \ldots, s, \quad \text{(if } s > 0)$$
$$\nu_{s+1} = h_{s+1},$$
$$\nu_j = \mu_{j-1}, j = s+2, \ldots, t, \quad \text{(if } t > s+1)$$
$$\nu_j = \mu_j, j = t+1, \ldots, k, \quad \text{(if } k > t).$$

In effect, μ_t is replaced by h_{s+1}, and the resulting μ_j's are re-indexed to restore the ordering. By (12), $E_r(\nu) = E_r(\mu)$. It is then a straightforward matter to verify that $\nu \in S_{k,s+1}(h)$. This completes the proof of the lemma.

We are now in a position to complete the proof of the theorem. If the eigenvalues of H are distinct, then for o.n. $x_1, \ldots, x_k$,

$$g(x_1, \ldots, x_k) \leqslant \max_{\lambda \in R_k(h)} E_r(\lambda)$$
$$= E_r(h_1, \ldots, h_s, h_{n-k+s+1}, \ldots, h_n).$$

for some s, $0 \leqslant s \leqslant k$. Now g attains this value for o.n. eigenvectors $y_1, \ldots, y_k$ corresponding to $h_1, \ldots, h_s, h_{n-k+s+1}, \ldots, h_n$, respectively. Thus

$$\max g = \max_{0 \leqslant s \leqslant k} E_r(h_1, \ldots, h_s, h_{n-k+s+1}, \ldots, h_n).$$

A similar result holds for the minimum. That these results remain valid when the eigenvalues of H are not all different follows by a continuity argument.

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