This is a ``preproof'' accepted article for *Canadian Mathematical Bulletin*
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DOI: 10.4153/S0008439525100829

A FREE BOUNDARY PROBLEM OF COMPETITION-DIFFUSION SYSTEM WITH DIRICHLET BOUNDARY CONDITION

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Abstract. In this paper, we investigate a free boundary problem for the Lotka-Volterra model consisting of an invasive species with density u and a native species with density v in one dimension. We assume that v undergoes diffusion and growth in $[0,+\infty)$, and u invades into the environment with spreading front $x = h(t)$ satisfying free boundary condition $h'(t) = -u_x(t, h(t)) - \alpha$ for some decay rate $\alpha > 0$, this is caused by the bad environment at the boundary. When u is an inferior competitor, $u(t, x)$ and $h(t)$ tend to 0 within a finite time, while another specie $v(t, x)$ tends to a stationary $\Lambda(x)$ defined on the half-line. When u is a superior competitor, we have a trichotomy result: spreading of u and vanishing of v (i.e., as $t \to +\infty$, $h(t)$ goes to $+\infty$ and $(u, v) \to (\Lambda, 0)$); the transition case (i.e., as $t \to +\infty$, $(u, v) \to (w_\alpha, \eta_\alpha)$, $h(t)$ tends to a finite point); vanishing of u and spreading of v (i.e., $u(t, x)$ and $h(t)$ tends to 0 within a finite time, $v(t, x)$ converges to $\Lambda(x)$). Additionally, we show that this trichotomy result depends on the initial data $u(0, x)$.

1. Introduction.

Consider the following free boundary problem

$$
\begin{cases}
u_t = u_{xx} + u(1 - u - k_1 v), & 0 < x < h(t),\ t > 0,\\
v_t = v_{xx} + v(1 - v - k_2 u), & 0 < x < +\infty,\ t > 0,\\
v(t,0) = 0,\ u(t,0) = u(t,h(t)) = 0, & t > 0,\\
h'(t) = -u_x(t,h(t)) - \alpha, & t > 0,\\
h(0) = h_0,\ \ u(0,x) = u_0(x), & 0 \le x \le h_0,\\
v(0,x) = v_0(x), & 0 \le x \le \infty,
\end{cases}
\tag{1.1}
$$

where $x = h(t)$ is a moving boundary, and $(u(t, x), v(t, x), h(t))$ is to be determined. h_0, α and $k_i (i = 1, 2)$ are given positive constants. The initial data v_0 satisfies

$$
v_0 \in C^2([0,+\infty)), v_0(0) = 0 \ \textit{and}\ v_0 \ge 0, \not\equiv 0 \ \textit{in}\ [0,+\infty),
\tag{1.2}
$$

and u_0 belongs to $\mathscr{X}(h_0)$ for some $h_0 > 0$, where

$$
\mathscr{X}(h_0) = \left\{\phi \in C^2([0,h_0]), \phi(0) = \phi(h_0) = 0,\ \phi > 0 \ \textit{in}\ [0,h_0)\right\}.
\tag{1.3}
$$

Ecologically, u is the density of an invasive species and v is the density of a native species density, k_i is interspecific competition rate. Free boundary $x = h(t)$ is the invading front of the invasive species, and it evolves according to the Stefan condition $h'(t) = -u_x(t, h(t)) - \alpha$. We use $\alpha > 0$ to denote the decay rate caused by the bad environment at (or, out of) the boundary such as the food, predators.

Key words and phrases. Lotka-Volterra model, advection, free boundary problem, asymptotic behavior.
2010 *Mathematics Subject Classification.* 35K57, 35B08, 35B40.
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When decay rate $\alpha = 0$ and left boundary condition is Neumann boundary condition, Du and Lin [7] studied the following free boundary problem

$$\begin{cases} u_t = d_1\Delta u + u(a_1 - b_1 u - c_1 v), & 0 < r < h(t),\ t > 0, \\ v_t = d_2\Delta v + v(a_2 - b_2 u - c_2 v), & 0 < r < +\infty,\ t > 0, \\ v_r(t,0) = 0,\ u_r(t,0) = u(t,h(t)) = 0, & t > 0, \\ h'(t) = -u_r(t,h(t)), & t > 0, \\ h(0) = h_0,\quad u(0,r) = u_0(r), & 0 \le r \le h_0, \\ v(0,r) = v_0(r), & 0 \le r \le \infty. \end{cases} \tag{1.4}$$

They used this problem to describe the dynamical process of a new competitor u who invades into the habitat of a native species v. The species u spreads through random diffusion in $[0, h(t)]$ with spreading front $h(t)$ satisfying classical Stefan condition $h'(t) = -u_r(t, h(t))$. They studied the long time behavior of species u and v, and obtained spreading-vanishing dichotomy. In problem (1.1), $a_1 = b_1 = a_2 = c_2 = 1$, $c_1 = k_1$ and $b_2 = k_2$. For the value of k_i, There are the following four cases (for more details, cf. [15,17]):

(1) $k_i \in (0,1), i = 1,2$;

(2) $0 < k_2 < 1 < k_1$;

(3) $0 < k_1 < 1 < k_2$;

(4) $k_i \in (1,\infty), i = 1,2$.

The case (1) is called the weak competition, in this case the competitors co-exist in the future. The case (2) and (3) is usually called weak-strong competition case, while (4) are known as the strong competition cases, but the asymptotic behavior of solutions is difficult to be given clearly. In [7], they considered the case (3) and (4).

There are some other competition systems with free boundaries, such as [10,20,22,24]. In [24], they researched the dynamics for a Lotka-Volterra type weak competition system with two free boundaries. Later, [10] investigated the spreading speeds and long time behavior of two invasive species (u, v), where (u, v) satisfies

$$\begin{cases} u_t = u_{xx} + u(1 - u - k_1 v), & 0 < x < h(t),\ t > 0, \\ v_t = v_{xx} + v(1 - v - k_2 u), & 0 < x < g(t),\ t > 0, \\ v_x(t,0) = 0,\ u_x(t,0) = u(t,h(t)) = 0, & t > 0, \\ h'(t) = -\mu u_x(t,h(t)),\ g'(t) = -\mu v_x(t,g(t)) & t > 0, \end{cases} \tag{1.5}$$

In [20], they investigated free boundary problems of the prey-predator model with two species living in $[0, h(t)]$, and they satisfy the condition $h'(t) = -\mu(u_x(t,x) + \rho v_x(t,x))$ at the same boundary $x = h(t)$. They considered the long time behavior of solution and criteria for spreading and vanishing.

In [22], they studied a predator-prey model with double free boundaries, two species satisfy

$$u_t = u_{xx} + u(1 - u + av), g(t) < x < h(t) \quad \text{and} \quad v_t = dv_{xx} + v(b - v - cu), x \in \mathbb{R},$$

with free boundary conditions

$$h'(t) = -\mu u_x(t,h(t)),\ g'(t) = -\mu u_x(t,g(t)).$$

They proved a spreading-vanishing dichotomy. Recently, in [26], the authors investigated a nonlocal diffusion competition model with seasonal succession and free boundaries. Yang [25] studied a competitive model by considering traveling wave solutions. Besides these, there are also other works concerning free boundary problem of diffusion systems (such as [14,18,21,23,28]).

If $v \equiv 0$, this means that there are no native species, the systems reduce to the following diffusive problem

$$\begin{cases} u_t = u_{xx} + u(1-u), & 0 < x < h(t),\ t > 0, \\ u(t,0) = u(t,h(t)) = 0, & t > 0, \\ h'(t) = -u_x(t,h(t)) - \alpha, & t > 0, \\ h(0) = h_0,\ \ u(0,x) = u_0(x), & 0 \le x \le h_0, \end{cases} \tag{1.6}$$

Such a similar problem is studied in [4] with decay rate $\alpha > 0$. It is more difficult for the solution to spread than the case $\alpha = 0$. Since $h'(t) > 0$ only if $u_x(t,h(t)) < -\alpha$. They obtained a different vanishing and a transition case. Also, [2, 3, 11] studied such problems with $\alpha > 0$ representing the decay rate caused by the bad environment at the boundary, such as food, predators and so on. Moreover, such a boundary condition are often used in the growth of protocell (cf. [12,27]), in the process of diffusion and polymerization of building materials, there is a disintegration (denoted by α) produced by many factors, such as aging. So the boundary condition satisfies

$$V_n = -\frac{\partial u}{\partial n} - \alpha,$$

where V_n is the velocity in the direction n, see Fig. 1. In one-dimensional space, this condition is our

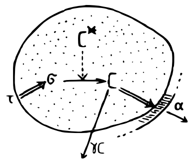

FIGURE 1. The growth of the protocell model.

condition used in the problem (1.1). Such a boundary condition is also used the tumour model.

In this paper, we will consider the spreading of two species u and v. There is new species u invading into an environment where a native competitor v already exists on the whole space, see Fig. 2. The

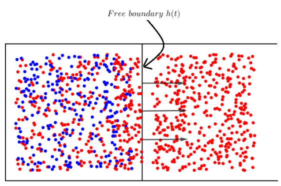

FIGURE 2. The spreading of two species u and v.

survival interval of the species v is $[0, +\infty]$, while the survival interval of u is $[0, h(t)]$. Since u is moving depending the food, the density of u and other factors, so the survival boundary of u, i.e., $h(t)$ is often depending on time t, there are three situations for the limits of $h(t)$, i.e., 0, $+\infty$ or a finite number, see Fig. 3.

The situations of spreading of these two species u and v are complicated, we only consider two cases for interspecific competition rate: $0 < k_2 < 1 < k_1$ and $0 < k_1 < 1 < k_2$; Usually, there is a decay rate at the boundary, we will analysis such decay rate α how to affect the spreading of two species. Besides, we will show that the initial data of u plays an crucial role in the asymptotic behavior of two species.

From [3,20,22], the problem (1.1) has a unique solution (u, v, h), with $v(t,x) \in C^{(1+\varrho)/2,1+\varrho}([0,+\infty) \times [0,+\infty))$, $u(t,x) \in C^{(1+\varrho)/2,1+\varrho}([0,T] \times [0,h(t)])$, $h(t) \in C^{1+\varrho/2}([0,T])$, where $\varrho \in (0,1)$, $T \in (0,+\infty]$. In

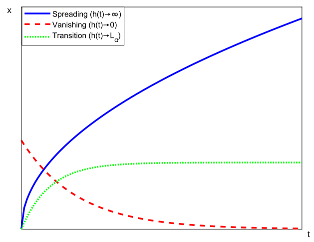

FIGURE 3. The limits of free boundary $h(t)$.

the following, we show that in some cases $T < +\infty$ and for other cases $T = +\infty$, this depends on the initial data. However, as in the proof of [4], the limit $h_T := \lim_{t\to T} h(t) \in [0, +\infty]$ exists. In particular, we also write h_T as h_∞ when $T = +\infty$.

In this paper, we have the following two main results:

Main result 1(Theorem 3.2). When $0 < k_2 < 1 < k_1$, we have the following results for the spreading of (u, v):

$$T < +\infty, \ \lim_{t\to T} h(t) = 0, \ \lim_{t\to T} \max_{x\in[0,h(t)]} u(t, x) = 0 \tag{1.7}$$

and

$$\lim_{t\to+\infty} v(t, x) = \Lambda(x) \quad \text{locally uniformly for } x \in [0, +\infty), \tag{1.8}$$

where $\Lambda(x)$ satisfies

$$\begin{cases} \Lambda'' + \Lambda(1 - \Lambda) = 0, & x > 0, \\ \Lambda(0) = 0, \ \Lambda(+\infty) = 1. \end{cases} \tag{1.9}$$

Remark 1.1. To explain the results of main results 1(i.e., Theorem 3.2), we give the numerical results by taking $k_1 = 2$, $k_2 = 0.3$ $\alpha = 0.2$ and $h_0 = 2$, see Fig. 4.

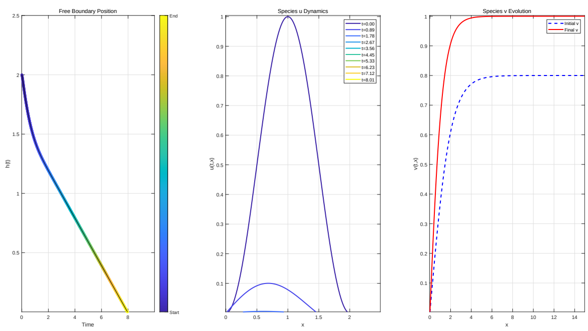

FIGURE 4. Numerical simulations of (u, v, h) are shown, from left to right, they are: the shrinking of free boundary $h(t)$ when u vanishes, vanishing of u, the spreading of v.

Main result 2 (Theorem 4.1). When $0 < k_1 < 1 < k_2$ and $0 < \alpha < \alpha_0 := \sqrt{3}/3$, we have, the solution (u, v, h) is either in

Case (I) : spreading of u and vanishing of v: $T = +\infty$, $\lim_{t\to+\infty} u(t, \cdot) = \Lambda(\cdot)$ and $\lim_{t\to+\infty} v(t, \cdot) = 0$ uniformly in any bounded subset of $[0, +\infty)$;

or

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Case (II): vanishing of u and spreading of v: (1.7) and (1.8) hold;
or
Case (III): in the transition case: $h_\infty = L_\alpha$, $u(t,\cdot) \to w_\alpha(\cdot)$ locally uniformly in $(0, h_\infty)$ and $v(t,\cdot) \to \eta_\alpha(\cdot)$ locally uniformly in $(0, +\infty)$, where $(w_\alpha, \eta_\alpha, L_\alpha)$ is the solution of

$$
\begin{cases}
-w'' = w(1 - w - k_1\eta), \quad 0 < x < +\infty,\\
-\eta'' = \eta(1 - \eta - k_2 w), \quad 0 < x < +\infty,\\
w(0) = w(L_\alpha) = 0, \ -w'(L_\alpha) = \alpha,\\
w(x) > 0 \ for \ x \in (0, L_\alpha),\\
w(x) \equiv 0 \ for \ x \geq L_\alpha,\\
\eta(0) = 0, \eta(+\infty) = 1.
\end{cases} \tag{1.10}
$$

Moreover, for the initial data $u_0 = \sigma\phi$ with $\phi \in \mathscr{X}(h_0)$, we prove that there is a critical data σ^* such that Case (I) holds when $\sigma > \sigma^*$, Case (II) holds when $\sigma < \sigma^*$, and Case (III) holds when $\sigma = \sigma^*$.

Remark 1.2. The existence of the solution $(w_\alpha, \eta_\alpha, L_\alpha)$ is given in Lemma 2.7. When $\alpha \geq \alpha_0$, there is no compactly solution.

We organized this paper as follows. In section 2, we give the general existence and uniqueness results, and the comparison principle. In section 3 and 4, we study the long time behavior of solutions when $0 < k_1 < k_2$ and $0 < k_2 < k_1$ respectively. In section 5, we give some sufficient conditions for spreading, vanishing and transition, and complete the proof main results.

2. Preliminary results.

In this section, we first prove a local existence and uniqueness result for the problem (1.1). Consider the following general free boundary problem

$$
\begin{cases}
u_t = u_{xx} + f(u, v), & 0 < x < h(t), \ t > 0,\\
v_t = v_{xx} + g(u, v), & 0 < x < +\infty, \ t > 0,\\
v(t, 0) = 0, \ u(t, 0) = u(t, h(t)) = 0, & t > 0,\\
h'(t) = -u_x(t, h(t)) - \alpha, & t > 0,\\
h(0) = h_0, \ \ u(0, x) = u_0(x), & 0 \leq x \leq h_0,\\
v(0, x) = v_0(x), & 0 \leq x \leq \infty,
\end{cases} \tag{2.1}
$$

where $f(0, v) = g(u, 0) = 0$ for any $u, v \in \mathbb{R}$, u_0 belongs to $\mathscr{X}(h_0)$ and v_0 satisfies (1.2).

Theorem 2.1. *Assume that f and g are locally Lipschitz continuous in $\mathbb{R}^2_+$. For any $\varrho \in (0,1)$, $u_0 \in \mathscr{X}(h_0)$ and v_0 satisfies* (1.2)*, there exists $T > 0$ such that problem* (2.1) *admits a unique bounded solution defined on* $[0, T)$ *with* $T \in [0, +\infty]$ *and*

$$
(u, v, h) \in C^{(1+\varrho)/2, 1+\varrho}(D_T) \times C^{(1+\varrho)/2, 1+\varrho}(D_T^\infty) \times C^{1+\varrho/2}([0, T]), \tag{2.2}
$$

moreover,

$$
||u||_{C^{(1+\varrho)/2,1+\varrho}(D_T)} + ||v||_{C^{(1+\varrho)/2,1+\varrho}(D_T^\infty)} + ||h||_{C^{1+\varrho/2,1+\varrho}([0,T])} \leq C,
$$

where $D_T = \{(t, x) \in \mathbb{R}^2 : t \in [0, T], \ x \in [0, h(t)]\}$, $D_T^\infty = \{(t, x) \in \mathbb{R}^2 : t \in [0, T], x \in [0, +\infty)\}$, C and T only depend on $h_0, \varrho, ||u_0||_{C^2([0,h_0])}, ||v_0||_{C^2([0,\infty))}$ and the local Lipschitz coefficients of f, g.

Theorem 2.2. *Under the assumptions of Theorem* 2.1*, if we assume further that there exists a constant $L > 0$ such that $f(u, v) \leq L(u + v)$ and $g(u, v) \leq L(u + v)$ for $u, v \geq 0$, then the unique solution $v(t, x)$ obtained in Theorem* 2.1 *can be extended uniquely to all $t > 0$, and if $\inf\limits_{0<t<T} h(t) > 0$, then the solution can be extended to a bigger interval $(0, T_*)$ with $T_* > T$.*

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Remark 2.3. We only consider the long time behaviour of bounded solutions, so conditions in Theorem 2.2 is used to exclude the possibility that (u, v) blows up in finite time. Of course, the problem (1.1) satisfies conditions in Theorem 2.2. Recall that we introduce $\alpha > 0$ in the free boundary condition. The property $h'(t) > 0$ in case $\alpha = 0$ (as shown in [4]) is no longer necessarily to be true. On the contrary, in some cases, the domain $[0, h(t)]$ may shrink, even, to a point.

By Theorem 2.1 and Theorem 2.2, we have the following estimates.

Theorem 2.4. *Problem* (1.1) *admits a unique and uniformly bounded solution* (u, v, h) *satisfying* (2.2). *And the solution* $v(t, x)$ *can be extended to all* $t > 0$ *and* $u(t, x)$ *is defined on* $[0, T)$ *with* $T \in (0, +\infty]$. *Moreover, there exist constants* M_1 *and* M_2 *such that*

$$0 < u(t, x) \leq M_1 \ \ for\, t \in [0, T),\ 0 \leq x < h(t),$$

$$0 < v(t, x) \leq M_2 \ \ for\, t \in [0, +\infty),\ 0 \leq x < +\infty.$$

And, there exists a constant M_3 *such that*

$$-\alpha < h'(t) \leq M_3 \ \ for\, t \in (0, T).$$

Proof. From Theorems 2.1-2.2, the problem (1.1) admits a unique solution (u, v, h) satisfying (2.2), and $v(t, x)$ is defined for all $t > 0$, $u(t, x)$ can be extended to $t \in [0, +\infty)$ as long as $h(t) > 0$ for $t > 0$.

It follows from the comparison principle that $u(t, x) \leq \bar{u}(t)$ for $t \in (0, T)$ and $x \in [0, h(t)]$, where

$$\bar{u}(t) := e^t \left(e^t - 1 + \frac{1}{\|u_0\|_\infty} \right)^{-1}$$

which is the solution of the problem

$$\begin{cases} \dfrac{d\bar{u}}{dt} = \bar{u}(1 - \bar{u}), \quad t > 0, \\ \bar{u}(0) = \|u_0\|_\infty. \end{cases} \tag{2.3}$$

Thus we have

$$u(t, x) \leq M_1 := \sup \bar{u}(t),\ t > 0.$$

Since $v(t, x)$ satisfies

$$\begin{cases} v_t - v_{xx} \geq v(1 - v), \quad t > 0,\ 0 \leq x < +\infty, \\ v(0, x) = v_0(x) \geq 0, \quad 0 \leq x < +\infty. \end{cases}$$

We have $v(t, x) \leq M_2 := max\{\|v_0\|_{L^\infty(0,+\infty)}, 1\}$.

Using the strong maximum principle to the equation of u we obtain

$$u(t, x) > 0\ for\ \ t \in (0, T),\ 0 < x < h(t),$$

and $v(t, x) > 0$ for $t > 0,\ 0 < x < \infty$.

Additionally, to estimate the boundness of $h(t)$, we construct the function

$$\widetilde{U}(t, x) := M_1[2M(h(t) - x) - M^2(h(t) - x)^2] \tag{2.4}$$

defined on

$$Q := \{(t, x) : 0 < t < \widetilde{T},\ max\{h(t) - M^{-1}, 0\} < x < h(t)\},$$

where

$$M := max\left\{ \frac{\alpha + \sqrt{\alpha^2 + 2}}{2}, \frac{4\|u_0\|_{C^1([-h_0,h_0])}}{3C_1} \right\}.$$

By a direct calculation, $0 \leq \widetilde{U} \leq M_1$ in Q. The definitions of $\widetilde{U}$ and M imply that

$$\widetilde{U}_t - \widetilde{U}_{xx} - \widetilde{U}(1 - \widetilde{U}) \geq C_1(2M^2 - 2M\alpha - 1) \geq 0\ \ in\ Q.$$

Additionally, when $h(t) \geq M^{-1}$,

$$\begin{cases} \widetilde{U}(t,h(t)) = u(t,h(t)) = 0, & t \in (0,\widetilde{T}), \\ u_0(x) \leq \widetilde{U}(0,x), & x \in [h_0 - M^{-1}, h_0] \cap [-h_0, h_0], \\ \widetilde{U}(t, h(t) - M^{-1}) = M_1 \geq u(t, h(t) - M^{-1}), \end{cases}$$

and

$$\widetilde{U}(t,0) > 0 = u(t,0).$$

Hence, it derives from the comparison principle that $u(t,x) \leq \widetilde{U}(t,x)$ in Q. Therefore,

$$h'(t) = -u_x(t,h(t)) - \alpha \leq U_x(t,h(t)) - \alpha = 2M_1 M - \alpha := M_3$$

□

For given a pair of functions $\underline{\mathbf{u}} := (\underline{u}, \underline{v})$ and $\overline{\mathbf{u}} := (\overline{u}, \overline{v})$, denote

$$[\overline{\mathbf{u}}, \underline{\mathbf{u}}] = \{\mathbf{u} := (u,v) \in [C([0,T] \times [0,+\infty))]^2 : (\underline{u}, \underline{v}) \leq (u,v) \leq (\overline{u}, \overline{v})\}.$$

For $(u_1, v_1) \leq (u_2, v_2)$, we mean $u_1 \leq u_2$ and $v_1 \leq v_2$.

We next give the comparison principle for general case which includes the problem (1.1). That is $f(u,v) = u(1 - u - k_1 v)$ and $g(u,v) = v(1 - v - k_2 u)$.

Lemma 2.5. *(The Comparison Principle). Let (f,g) be quasimonotone nonincreasing and Lipschitz continuous in $[\overline{\mathbf{u}}, \underline{\mathbf{u}}]$, with $f(0,v) = g(u,0) = 0$. Assume that $T \in [0,+\infty)$, $\underline{h}, \overline{h} \in C^1([0,T])$,*

$$\underline{u} \in C(\overline{\Sigma_T^1}) \cap C^{1,2}(\Sigma_T^1) \text{ with } \Sigma_T^1 := \{(t,x) \in \mathbb{R}^2 : t \in (0,T],\ x \in (0, \underline{h}(t))\},$$

$$\overline{u} \in C(\overline{\Sigma_T^2}) \cap C^{1,2}(\Sigma_T^2) \text{ with } \Sigma_T^2 := \{(t,x) \in \mathbb{R}^2 : t \in (0,T],\ x \in (0, \overline{h}(t))\},$$

$$\underline{v}, \overline{v} \in (L^\infty \cap C)([0,T] \times [0,\infty)) \cap C^{1,2}((0,T] \times [0,+\infty)),$$

and $\underline{u}, \overline{u}, \underline{v}, \overline{v} \geq 0$, $h(0) > 0$, such that

$$\begin{cases} \overline{u}_t - \overline{u}_{xx} \geq f(\overline{u}, \underline{v}), & 0 < t \leq T,\ 0 \leq x < \overline{h}(t), \\ \underline{u}_t - \underline{u}_{xx} \leq f(\underline{u}, \overline{v}), & 0 < t \leq T,\ 0 \leq x < \underline{h}(t), \\ \overline{v}_t - \overline{v}_{xx} \geq g(\underline{u}, \overline{v}), & 0 < t \leq T,\ 0 \leq x < +\infty, \\ \underline{v}_t - \underline{v}_{xx} \leq g(\overline{u}, \underline{v}), & 0 < t \leq T,\ 0 \leq x < +\infty, \\ \overline{u}(t,0) = \underline{v}(t,0) = 0,\ \overline{u}(t,x) = 0, & 0 < t \leq T,\ \overline{h}(t) \leq x < +\infty, \\ \underline{u}(t,0) = \overline{v}(t,0) = 0,\ \underline{u}(t,x) = 0, & 0 < t \leq T,\ \underline{h}(t) \leq x < +\infty, \\ \underline{h}'(t) \leq -\underline{u}_x(t,h(t)) - \alpha, & 0 < t \leq T, \\ \overline{h}'(t) \geq -\overline{u}_x(t,h(t)) - \alpha, & 0 < t \leq T, \\ \underline{h}(0) \leq h_0 \leq \overline{h}(0), & \\ \underline{u}_0(x) \leq u_0(x) \leq \overline{u}_0(x), & 0 \leq x \leq h_0, \\ \underline{v}_0(x) \leq v_0(x) \leq \overline{v}_0(x), & 0 \leq x \leq +\infty. \end{cases} \tag{2.5}$$

Let (u,v,h) be the unique bounded solution of (2.1) *with initial data (u_0, v_0). Then $\underline{h}(t) \leq h(t)) \leq \overline{h}(t)$ in $(0,T]$, $\underline{u}(t,x) \leq u(t,x) \leq \overline{u}(t,x)$ for $(t,x) \in (0,T] \times [0,h(t))$ and $\underline{v}(t,x) \leq v(t,x) \leq \overline{v}(t,x)$ for $(t,x) \in (0,T] \times [0,+\infty)$.*

Remark 2.6. If α in $(2.5)_7$ is replaced by some $\beta > \alpha$, or/and α in $(2.5)_8$ is replaced by some $\gamma < \alpha$, the conclusions of Lemma 2.5 are also true. We call $(\overline{u}, \underline{v})$ (resp. $(\underline{u}, \overline{v})$) is the upper solution (resp. lower solution).

Lemma 2.7. *Assume $0 < k_1 < 1 < k_2$ and $0 < \alpha < \alpha_0 := \sqrt{3}/3$, the problem (1.10) has a solution $(w_\alpha, \eta_\alpha, L_\alpha)$.*

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Proof. By the phase plane analysis (cf. [3]), there exists L_1^*, for any $L > L_1^*$, the problem

$$\begin{cases} -\bar{w}'' = \bar{w}(1-\bar{w}), & 0 < x < L, \\ \bar{w}(0) = \bar{w}(L) = 0 \end{cases} \tag{2.6}$$

has a unique compactly supported solution, denoted by $w_c^{(1)}$, and $-w_c^{(1)}(L) > 0$, $-w_c^{(1)}(L) < \alpha_0 := -w'_\infty(0)$, where w_∞ is the solution of

$$\begin{cases} -w'' = w(1-w), & 0 < x < +\infty, \\ w(0) = 0, \ w(+\infty) = 1. \end{cases} \tag{2.7}$$

Actually, from [2, Theorem 2.5], we have $w'_\infty(0) = \sqrt{3}/3$, for any $\alpha \in (0, \alpha_0)$, there is a l_α such that the problem (2.6) has a solution with $-\bar{w}'(l_\alpha) = \alpha$. And there is no compactly solution when $\alpha \geq \alpha_0$.

Also, there exists L_2^*, for any $L > L_2^*$, the problem

$$\begin{cases} -\underline{w}'' = \underline{w}(1-\underline{w}-k_1), & 0 < x < L, \\ \underline{w}(0) = \underline{w}(L) = 0, \end{cases} \tag{2.8}$$

has a unique compactly supported solution, denoted by $w_c^{(2)}$. Denote the solution of

$$\begin{cases} -\bar{\eta}'' = \bar{\eta}(1-\bar{\eta}), & 0 < x < +\infty, \\ \bar{\eta}(0), \ \bar{\eta}(+\infty) = 1, \end{cases} \tag{2.9}$$

as $\bar{\eta}(x)$. By the standard upper and lower solutions method, there is L_0^*, for $l > L_0^*$ the problem

$$\begin{cases} -w'' = w(1-w-k_1\eta), & x > 0, \\ -\eta'' = \eta(1-\eta-k_2 w), & x > 0, \\ w(0) = w(l) = \eta(0) = 0, \ w(x) > 0, \ 0 < x < l, \\ w(x) \equiv 0, x \geq l \end{cases} \tag{2.10}$$

has at least one positive solution, denoted by (w_l, η_l) and

$$w_c^{(2)}(x) \leq w_l(x) \leq w_c^{(1)}(x), \quad 0 \leq \eta_l(x) \leq \bar{\eta}(x), \quad 0 < x < l.$$

Moreover, $w'_l(0) > 0$ $-w'_l(l) > 0$, and $w'_l(l) \to 0$ as $l \to L_0^*$.

For any small $\varepsilon > 0$, (w_l, η_l) also satisfies

$$\begin{cases} -w'' = w(1-w-k_1\eta), & 0 < x < l, \\ -\eta'' = \eta(1-\eta-k_2 w), & 0 < x < l, \\ w(0) = w(l) = \eta(0) = 0, \ w(x) > 0, \ 0 < x < l, \\ w(x) \equiv 0, x \geq l, \ \eta(l) \leq 1+\varepsilon. \end{cases} \tag{2.11}$$

Letting $l \to +\infty$, then $(w_l, \eta_l) \to (w_\infty, 0)$, where w_∞ is the solution of (2.7), and $w'_\infty(0) = \sqrt{3}/3$.

For any $\alpha \in (0, \alpha_0)$, there is $l = L_\alpha$ such that the problem (2.10) has a solution satisfying $-w'(L_\alpha) = \alpha$. Denote such solution as $(w_\alpha, \eta_\alpha, L_\alpha)$, that is, the solution of the problem (1.10). □

3. Invasion of an inferior competitor.

In this section, we consider the situation that $0 < k_2 < 1 < k_1$, namely u is an inferior competitor. In this case, we analyze the asymptotic behavior of (u, v) and show that the inferior invader vanishes at last while the superior species always survives the invasion.

Lemma 3.1. *Let* (u, v, h) *be a solution of the problem* (1.1). (u, h) *is defined on* $[0, \widetilde{T})$ *with* $\widetilde{T} \in (0, +\infty]$. *If* $\lim_{t\to\widetilde{T}} h(t) = 0$*, then* $\widetilde{T} < +\infty$ *and*

$$\lim_{t\to\widetilde{T}} \max_{0\leq x\leq h(t)} u(t,x) = 0.$$

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Proof. From Theorem 2.4, we have $u(t,x) \le M_1$ for all $x \in [0,h(t)]$ and $t \in [0,T]$. Using the upper solution $\widetilde{U}(t,x)$ constructed in Theorem 2.4. Note that $\lim_{t\to\widetilde{T}} h(t) = 0$, then there exists $T_1 < \widetilde{T}$ such that $h(t) - M^{-1} < 0$ for $t > T_1$. Therefore $u(t,x) \le \widetilde{U}(t,x)$ for $t > T_1$ and $x \in [0,h(t)]$. Thus we have

$$\widetilde{U}(t,x) \le C_1[2Mh(t) - M^2h^2(t)] \to 0 \ \ as\ t \to \widetilde{T}.$$

From this and $u(t,x) \le \widetilde{U}(t,x)$, we have

$$\|u(t,\cdot)\|_{L^\infty([0,h(t)])} \to 0 \ \ as\ t \to \widetilde{T}. \tag{3.1}$$

We now prove that $\widetilde{T} < +\infty$ and $\lim_{t\to+\widetilde{T}} h(t) = 0$. For $\varepsilon_* \le \frac{\alpha}{4M}$, by (3.1), there exists $0 < T_2 < \widetilde{T}$ such that and $h(t) - M^{-1} < 0$ and $u(t,x) \le \varepsilon_*$. The function

$$\widetilde{U}_2(t,x) := \varepsilon_*[2M(h(t)-x) - M^2(h(t)-x)^2],\ \ x \in [0,h(t)]$$

is also an upper solution for $u(t,x)$. Therefore $u(t,h(t)) \le \widetilde{U}_2(t,x)$ for $x \in [0,h(t)]$ and $t > T_2$. Hence we have

$$-u_x(t,h(t)) \le -\left(\widetilde{U}_2\right)_x(t,h(t)) = 2M\varepsilon_* \le \frac{\alpha}{2}.$$

Hence $h'(t) = -u_x(t,h(t)) - \alpha \le -\frac{\alpha}{2}$, this means that $h(t) \to 0$ as $t \to \widetilde{T} \le \frac{2h_0}{\alpha}$. □

Theorem 3.2. *Let* (u,v,h) *be the solution of* (1.1), (u,h) *defined on* $[0,T)$ *with* $T \in [0,+\infty]$, *if* $0 < k_2 < 1 < k_1$ *and* $v_0 \not\equiv 0$, *then* u *vanishes:*

$$T < +\infty,\ \lim_{t\to T} \max_{x\in[0,h(t)]} u(t,x) = 0,\ \lim_{t\to T} h(t) = 0,$$

and v *spreads:*

$$\lim_{t\to+\infty} v(t,x) = \Lambda(x)\ locally\ uniformly\ for\ x \in [0,+\infty),$$

where $\Lambda(x)$ *satisfies* (1.9).

Proof. We assume $T = +\infty$. From Theorem 2.4, we have $u(t,x) \le \bar{u}(t)$ for $t > 0$ and $x \in [0,h(t)]$. Since $\lim_{t\to+\infty} \bar{u}(t) = 1$, we deduce that

$$\limsup_{t\to+\infty} u(t,x) \le 1\ uniformly\ for\ x \in [0,\infty).$$

Furthermore, consider the parabolic problem

$$\begin{cases} \Phi_t - \Phi_{xx} = \Phi(1-\Phi), & t > 0,\ x > 0, \\ \Phi(t,0) = 0, & t > 0, \\ \Phi(0,x) = u_0(x), & x > 0. \end{cases} \tag{3.2}$$

Letting $t \to +\infty$ we have

$$\Phi(t,\cdot) \to \Lambda(\cdot)\ in\ C^2[0,+\infty], \tag{3.3}$$

where $\Lambda(\cdot)$ satisfies (1.9). From the problem (1.1), we get that

$$u_1 = u_{xx} + u(1-u-k_1v) \le u_{xx} + u(1-u),\ \text{and}\ u(t,0) = 0, u(t,h(t)) = 0.$$

So $\Phi(t,x)$ is upper solution of $u(t,x)$. Hence $u(t,x) \le \Phi(t,x)$ for $t > 0$ and $x \in [0,h(t)]$. Combining this and (3.3), we deduce

$$\limsup_{t\to+\infty} u(t,x) \le \Lambda(x)\ uniformly\ for\ x \in [0,+\infty).$$

Similarly, we have

$$\limsup_{t\to+\infty} v(t,x) \le \Lambda(x) \le 1\ uniformly\ for\ x \in [0,+\infty). \tag{3.4}$$

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Therefore for any small $\varepsilon_1 < \frac{1}{k_2} - 1$, there exist $t_1 > 0$ such that $u(t,x) \le 1 + \varepsilon_1$ for $t \ge t_1, x \in [0,\infty)$. Then v satisfies

$$\begin{cases} v_t - v_{xx} \ge v(1 - k_2 - k_2\varepsilon_1 - v), & t > t_1, 0 \le x < +\infty, \\ v(t,0) = 0, & t > t_1, \\ v(t_1,x) > 0, & 0 \le x < +\infty. \end{cases} \tag{3.5}$$

Let v_* be the unique solution of

$$\begin{cases} (v_*)_t - (v_*)_{xx} = v_*(1 - k_2 - k_2\varepsilon_1 - v_*), & t > t_1, 0 \le x < +\infty, \\ (v_*)(t,0) = 0, & t > t_1, \\ (v_*)(t_1,x) = v(t_1,x), & 0 \le x < +\infty. \end{cases}$$

By the comparison principle, we have $v(t,x) > v_*(t,x)$ for all $t > t_1$ and $x \ge 0$. From [8], we obtain $\lim_{t\to\infty} v_*(t,\cdot) = \tilde{v}_*(\cdot)$ uniformly in any bounded subset of $[0,\infty)$, where $\tilde{v}_*$ satisfies

$$\begin{cases} \tilde{v}_*'' + \tilde{v}_*(1 - k_2 - k_2\varepsilon_1 - \tilde{v}_*) = 0, \ for\ x > 0, \\ \tilde{v}_*(0) = 0, \tilde{v}_*(+\infty) = 1 - k_2 - k_2\varepsilon_1, \\ \tilde{v}_*(x) > 0, \ for\ x > 0. \end{cases}$$

Therefore, for any large $L > 0$, there exists $t_L > t_1$ such that

$$v(t,x) \ge v_*(t,x) \ge \frac{\tilde{v}_*(x)}{2} \ for\ t \ge t_L, \ 0 \le x \le L. \tag{3.6}$$

Then (u,v) satisfies

$$\begin{cases} u_t = u_{xx} + u(1 - u - k_1 v), & 0 < x < h(t), \ t > t_L, \\ v_t = v_{xx} + v(1 - v - k_2 u), & 0 < x < +\infty, \ t > t_L, \\ v(t,0) = 0, \ u(t,0) = u(t,h(t)) = 0, & t > t_L, \\ u(t,x) \le 1 + \varepsilon_1, v(t,x) \ge \frac{\tilde{v}_*(x)}{2}, & 0 \le x \le L, t > t_L. \end{cases} \tag{3.7}$$

Since $u \equiv 0$ for $t > t_L, \ x \ge h(t)$, no matter whether or not $h(t) \le L$, we always have $u \le \bar{u}$ and $v \ge \underline{v}$ in $[t_L,\infty) \times [0,L]$, where $(\bar{u},\underline{v})$ satisfies

$$\begin{cases} \bar{u}_t = \bar{u}_{xx} + \bar{u}(1 - \bar{u} - k_1\underline{v}), & 0 < x < L, \ t > t_L, \\ \underline{v}_t = \underline{v}_{xx} + \underline{v}(1 - \underline{v} - k_2\bar{u}), & 0 < x < L, \ t > t_L, \\ \underline{v}(t,0) = 0, \ \bar{u}(t,0) = \bar{u}(t,h(t)) = 0, & t > t_L, \\ \bar{u}(t,x) = 1 + \varepsilon_1, \underline{v}(t,x) = \frac{\tilde{v}_*(x)}{2}, & 0 \le x \le L, \ x = L, \ or\ t = t_L. \end{cases} \tag{3.8}$$

The system (3.8) is quasimonotone nonincreasing, which generates a monotone dynamical system with respect to the order

$$(u_1,v_1) \le_\varsigma (u_2,v_2) \ if\ and\ only\ if\ u_1 \le u_2\ and\ v_1 \ge v_2.$$

Obviously, the initial value $(1 + \varepsilon_1, \frac{\tilde{v}_*}{2})$ is also an upper solution. By the theory of monotone dynamical systems (cf. [19]), we have $\lim_{t\to\infty} \bar{u}(t,x) = \bar{u}_L(x)$ and $\lim_{t\to\infty} \underline{v}(t,x) = \underline{v}_L(x)$ uniformly in $[0,L]$, where $(\bar{u},\underline{v})$ satisfies

$$\begin{cases} -(\bar{u}_L)_{xx} = \bar{u}_L(1 - \bar{u}_L - k_1\underline{v}_L), & 0 \le x < L, \\ -(\underline{v}_L)_{xx} = \underline{v}_L(1 - \underline{v}_L - k_2\underline{u}_L), & 0 \le x < L, \\ \bar{u}_L(0) = \underline{v}_L(0) = 0, \\ \bar{u}_L(L) = 1 + \varepsilon_1, \underline{v}_L(L) = \frac{\tilde{v}_*(L)}{2}, \end{cases} \tag{3.9}$$

and $(\bar{u}_L,\underline{v}_L) \le_\varsigma (1 + \varepsilon_1, \frac{\tilde{v}_*}{2})$.

Assume $0 < L_1 < L_2$, by comparing the boundary conditions and initial condition in (3.8) with L replaced by L_i $(i = 1,2)$, we have $\bar{u}_{L_1}(x) \ge \bar{u}_{L_2}(x)$ and $\underline{v}_{L_1}(x) \le \underline{v}_{L_2}(x)$ for $x \in [0,L_1]$.

Let $L \to +\infty$, then $(\bar{u}_L(x), \underline{v}_L(x)) \to (\bar{u}_\infty(x), \underline{v}_\infty(x))$, which satisfies

$$\begin{cases} -(\bar{u}_\infty)_{xx} = \bar{u}_\infty(1 - \bar{u}_\infty - k_1\underline{v}_\infty), & x \geq 0, \\ -(\underline{v}_\infty)_{xx} = \underline{v}_\infty(1 - \underline{v}_\infty - k_2\underline{u}_\infty), & x \geq 0, \\ \bar{u}_\infty(0) = \underline{v}_\infty(0) = 0, \\ \bar{u}_\infty(x) \leq 1 + \varepsilon_1, \underline{v}_\infty(x) \geq \frac{\tilde{v}_*(x)}{2}, & x \geq 0. \end{cases}$$

Next we show that $\bar{u}_\infty \equiv 0$ and $\underline{v}_\infty \equiv \Lambda(x)$. Let $(Z(t,x), W(t,x))$ be the solution of the problem

$$\begin{cases} Z_t - Z_{xx} = Z(1 - Z - k_1 W), & t > 0, x \geq 0, \\ W_t - W_{xx} = W(1 - W - k_2 Z), & t > 0, x \geq 0, \\ Z(t,0) = W(t,0) = 0, & t > 0, \\ Z(0,x) = 1 + \varepsilon_1, W(0,x) = \frac{\tilde{v}_*(x)}{2}, & x \geq 0. \end{cases} \tag{3.10}$$

From [16], we have $(Z, W) \to (0, \Lambda)$ as $t \to \infty$ uniformly in $[0, \infty)$. By the comparison principle, we get that $\bar{u}_\infty(x) \leq Z(t,x)$ and $\underline{v}_\infty(x) \geq W(t,x)$ for $t > 0$, which gives that $\bar{u}_\infty(x) \equiv 0$. Combining this with (3.4) we have $\underline{v}_\infty = \Lambda$.

Thus we have $\limsup_{t\to+\infty} u(t,\cdot) \leq 0$ and $\liminf_{t\to+\infty} v(t,\cdot) \geq \Lambda(\cdot)$ uniformly in $[0, L]$, which implies that $\lim_{t\to+\infty} u(t,\cdot) = 0$ and $\lim_{t\to+\infty} v(t,\cdot) = \Lambda(\cdot)$ uniformly in any bounded subset of $[0, +\infty)$.

However, by the proof of Lemma 3.1, $\widetilde{U}_2$ is also an upper solution, we can prove that $h(t) \to 0$ as $t \to \widehat{T}$ for some $\widehat{T} < +\infty$. This contradicts our assumption $T = +\infty$. Hence there must be $T < +\infty$, combining this and Theorem 2.2 we have $h(t) \to 0$ as $t \to T$. Additionally, by Lemma 3.1 we have $\lim_{t\to T} \max_{x\in[0,h(t)]} u(t,x) = 0$. □

4. Invasion of a superior competitor

In this section, we are devoted to the case that u is a superior competitor, that is $0 < k_1 < 1 < k_2$, we will have a trichotomy result (see Main result 2 in the introduction).

Theorem 4.1. *Assume $0 < \alpha < \frac{\sqrt{3}}{3}$. Let (u, v, h) be the unique solution of the problem (1.1), (u, h) defined on $[0, T)$ with $T \in (0, +\infty]$. Then the solution is either in Case (I) or Case (II) or Case (III) in the Main results 2 (see them in the introduction).*

We prove Theorem 4.1 by several lemmas (see the following Lemmas 4.2- 4.4). In the rest of this section, we always assume that (u, v, h) is the unique solution of (1.1) with $u_0 \in \mathscr{X}(h_0)$ and v_0 satisfies (1.2).

Lemma 4.2. *If $h_\infty = +\infty$, then $\lim_{t\to+\infty} u(t,\cdot) = \Lambda(\cdot)$ and $\lim_{t\to+\infty} v(t,\cdot) = 0$ uniformly in any bounded subset of $[0, +\infty)$, where $\Lambda(\cdot)$ satisfies (1.9).*

Proof. From the proof of Theorem 3.2, we have

$$\limsup_{t\to+\infty} u(t,x) \leq \Lambda(x) \leq 1, \quad \limsup_{t\to+\infty} v(t,x) \leq \Lambda(x) \leq 1 \tag{4.1}$$

uniformly for $x \in [0, +\infty)$, where $\Lambda(x)$ satisfies (1.9). Therefore, for any small positive $\varepsilon_2 < \frac{1-k_1}{k_1}$, there exists $t_2 > 0$ such that $v(t,x) \leq 1 + \varepsilon_2$ for $t \geq t_2, x \in [0, +\infty)$.

On the other hand, since $h_\infty = +\infty$, for any large $l > \pi\sqrt{\frac{1}{1-k_1-k_1\varepsilon_2}}$, there exists $t_l > t_2$ such that $h(t) > l$ for $t \geq t_l$. Then u satisfies

$$\begin{cases} u_t - u_{xx} \geq u(1 - u - k_1(1 + \varepsilon_2)), & 0 < x < h(t),\ t > t_l, \\ u(t,0) = u(t,h(t)) = 0, & t > t_l, \\ h'(t) = -u_x(t,h(t)) - \alpha, & t > t_l. \end{cases} \tag{4.2}$$

Consider the following problem,

$$\begin{cases} \underline{u}_t - \underline{u}_{xx} = \underline{u}(1 - \underline{u} - k_1(1 + \varepsilon_2)), & 0 < x < l,\ t > 0, \\ \underline{u}(t, 0) = \underline{u}(t, l) = 0, & t > 0, \\ \underline{u}(t_l, x) \le u(t_l, x), & 0 \le x \le l. \end{cases} \tag{4.3}$$

Denote its solution by $u_l^*(t, x)$, it follows from the comparison principle that $u(t, x) \ge u_l^*(t, x)$ for all $t > t_l$ and $x \in [0, l]$. Thus $u(t, x) \ge \frac{u_l^*(t,x)}{2}$ for $x \in [0, l]$ and $t > t_l$. Moreover,

$$\liminf_{t\to+\infty} u(t, x) \ge \lim_{t\to+\infty} \frac{u_l^*(t, x)}{2} \ge \frac{\widehat{u}_0(x)}{2},$$

where $\widehat{u}_0(x)$ is the nonegative solution of

$$\begin{cases} -q'' = q(1 - q - k_1(1 + \varepsilon_2)), & 0 < x < l, \\ q(0) = q(l) = 0. \end{cases} \tag{4.4}$$

By the phase plane analysis (cf, [3]), as $l \to +\infty$, $\widehat{u}_0(x)$ tends to $\Lambda(x)$ locally uniformly in $[0, +\infty)$. Additionally, (u, v) satisfies

$$\begin{cases} u_t - u_{xx} = u(1 - u - k_1 v), & 0 < x < l,\ t > t_l, \\ v_t - v_{xx} = v(1 - v - k_2 u), & 0 < x < l,\ t > t_l, \\ u(t, 0) = v(t, 0) = 0, & t > 0, \\ u(t, x) \ge \frac{u_l^*(t,x)}{2},\ v(t, x) \le \Lambda(x), & 0 \le x \le l,\ t \ge t_l. \end{cases} \tag{4.5}$$

Of course

$$\begin{cases} u_t - u_{xx} = u(1 - u - k_1 v), & 0 < x < l/2,\ t > t_l, \\ v_t - v_{xx} = v(1 - v - k_2 u), & 0 < x < l/2,\ t > t_l, \\ u(t, 0) = v(t, 0) = 0, & t > 0, \\ u(t, x) \ge \frac{u_l^*(t,x)}{2},\ v(t, x) \le \Lambda(x), & 0 \le x \le l/2,\ t \ge t_l. \end{cases} \tag{4.6}$$

As in the proof of Theorem 3.2, from the theory of monotone dynamical systems that

$$\liminf_{t\to+\infty} u(t, x) \ge \underline{u}_l(x)$$

and

$$\limsup_{t\to+\infty} v(t, x) \le \overline{v}_l(x)$$

in $[0, l/2]$, where $(\underline{u}_l, \overline{v}_l)$ satisfies

$$\begin{cases} -\underline{u}_l'' = \underline{u}_l(1 - \underline{u}_l - k_1\overline{v}_l), & 0 \le x < l/2, \\ -\overline{v}_l'' = \overline{v}_l(1 - \overline{v}_l - k_1\underline{u}_l), & 0 \le x < l/2, \\ \underline{u}_l(0) = \overline{v}_l(0) = 0, & \\ \underline{u}_l(l/2) = \frac{\widehat{u}_0(l/2)}{2},\ \overline{v}_l(l/2) = 1 + \varepsilon_2. & \end{cases} \tag{4.7}$$

Letting $l \to +\infty$, we have $(\underline{u}_l, \overline{v}_l) \to (\underline{u}_\infty, \overline{v}_\infty)$, where $(\underline{u}_\infty, \overline{v}_\infty)$ satisfies

$$\begin{cases} -\underline{u}_\infty'' = \underline{u}_\infty(1 - \underline{u}_\infty - k_1\overline{v}_\infty), & 0 \le x < +\infty, \\ -\overline{v}_\infty'' = \overline{v}_\infty(1 - \overline{v}_\infty - k_2\underline{u}_\infty), & 0 \le x < +\infty, \\ \underline{u}_\infty(0) = \overline{v}_\infty(0) = 0, & \\ \underline{u}_\infty(+\infty) \ge \lim_{l\to+\infty} \frac{\widehat{u}_0(l)}{2},\ \overline{v}_\infty(x) \le \Lambda, & 0 \le x < +\infty. \end{cases} \tag{4.8}$$

By the global dynamical behavior of the ODE system (cf. [16]), there hold $\underline{u}_\infty(x) = \Lambda(x)$ and $\overline{v}_\infty(x) = 0$. Therefore, $\liminf_{t\to+\infty} u(t, x) \ge \Lambda(x)$ and $\limsup_{t\to+\infty} v(t, x) \le 0$. By this and (4.1), we get

$$\lim_{t\to+\infty} u(t, \cdot) = \Lambda(\cdot) \quad \text{and} \quad \lim_{t\to+\infty} v(t, \cdot) = 0 \text{ locally uniformly in } [0, +\infty).$$

□

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By Lemma 3.1 and Theorem 3.2 we have the following results.

Lemma 4.3. *Let* (u, v, h) *be a solution of* (1.1). *If* $\lim_{t\to\widetilde{T}} h(t) = 0$, *then* $\widetilde{T} < +\infty$ *and*

$$\lim_{t\to\infty} \max_{0\le x\le h(t)} u(t,x) = 0,$$

$$\lim_{t\to+\infty} v(t,x) = \Lambda(x) \text{ locally uniformly for } x \in [0,+\infty),$$

where $\Lambda(x)$ *satisfies* (1.9).

Lemma 4.4. *Assume that* $0 < \alpha < \alpha_0$, *Let* (u, v, h) *be a solution of* (1.1). *If* $0 < h_\infty < +\infty$, *then* $h_\infty = L_\alpha$ *and* $(u, v) \to (w_\alpha, \eta_\alpha)$, *where* $(w_\alpha, \eta_\alpha, L_\alpha)$ *satisfies* (1.10).

Proof. By Theorem 2.1, for any sequence $\{t_n\}$, there exists subsequence t_{n_j} such that $u(t_{n_j},\cdot) \to w_0(\cdot)$ and $v(t_{n_j},\cdot) \to \eta_0(\cdot)$ as $t_{n_j} \to +\infty$, where (w_0, η_0) satisfies

$$\begin{cases} -w_0'' = w_0(1 - w_0 - k_1\eta_0), & 0 < x < +\infty, \\ -\eta_0'' = \eta_0(1 - \eta_0 - k_2 w_0), & 0 < x < +\infty, \\ w_0(0) = w_0(h_\infty) = 0, & \\ w_0(x) > 0 \ for \ x \in (0, h_\infty), & \\ w(x) \equiv 0 \ when \ x \ge h_\infty. & \end{cases} \tag{4.9}$$

For $\gamma \in (0,1)$, by passing to a subsequence,

$$\lim_{j\to\infty} \|u(\tilde{t}_{n_j},\cdot) - w_0(\cdot)\|_{C^{1+\gamma}([0,h(\tilde{t}_{n_j})])} \to 0 \text{ and } \lim_{j\to\infty} \|v(\tilde{t}_{n_j},\cdot) - \eta_0(\cdot)\|_{C^{1+\gamma}([0,h(\tilde{t}_{n_j})])} \to 0. \tag{4.10}$$

Therefore,

$$h'(t_{n_j}) = -u_x(t_{n_j}, h(t_{n_j})) - \alpha \to -w_0'(h_\infty) - \alpha > 0.$$

On the other hand, since $h_\infty \in (0,+\infty)$ and $h(t)$ is Hölder continuous when $h(t) > 0$. So $h'(t) \to 0$ as $t \to +\infty$, this implies that $-w_0'(h_\infty) = \alpha$. By uniqueness of the solution for the problem (4.9), we derive that (w_0, η_0) is nothing but (w_α, η_α) and $h_\infty = L_\alpha$. □

Lemma 4.5. *Suppose that* $\alpha > \alpha_0 = \frac{\sqrt{3}}{3}$, (u, v, h) *is a solution of the problem* (1.1), *with* (u, h) *defined on some maximal existence interval* $[0, \widetilde{T})$ *and* v *defined on* $[0, +\infty)$, *then*

$$\widetilde{T} < +\infty, \ \lim_{t\to+\widetilde{T}} h(t) = 0, \ \lim_{t\to\widetilde{T}} \max_{x\in[0,h(t)]} u(t,x) = 0,$$

and

$$\lim_{t\to+\infty} v(t,\cdot) = \Lambda(\cdot) \ locally \ uniformly \ in \ [0,+\infty).$$

Proof. Consider the following ODE problem,

$$\begin{cases} \zeta'' + \zeta(1-\zeta) = 0, & -\infty < x \le 0, \\ \zeta(0) = 0, \ \zeta(-\infty) = 1, & \\ \zeta(x) > 0 \ for \ x < 0. & \end{cases} \tag{4.11}$$

By the Hopf Lemma, $\zeta'(0) < 0$. Also $-\zeta'(0) = \frac{\sqrt{3}}{3}$.

We extend ζ to $[0,+\infty)$ by assuming

$$\zeta(x) = \zeta'(0)x \ for \ x > 0.$$

Define $\varepsilon_0 := \max_{0\le x\le h_0} u_0(x)$. Choose a sufficiently small $\varepsilon_3 > 0$, then there exists $\delta > 0$ such that

$$u(1-u) - (u+\varepsilon_3)(1-u-\varepsilon_3) \ge \delta\varepsilon_3 \ for \ 1-\varepsilon_3 \le u \le 1 \ and \ 0 \le \varepsilon_3 \le \varepsilon_0,$$

Define

$$\kappa := \max_{\zeta\le 1-\varepsilon_3} \zeta'(\xi) < 0,$$

https://doi.org/10.4153/S0008439525100829 Published online by Cambridge University Press

$$\nu := (-\zeta'(0) - \alpha)\varepsilon_0^{-1} < 0,$$
$$M := max\left\{\frac{\delta\kappa}{\zeta'(0)}, \delta + 1\right\}.$$

Construct an upper soltuion $\overline{U}(t, x) := \zeta(x - x^* + \xi(t)) + q(t)$, for $t > 0$ and $0 \le x \le \overline{h}(t)$, where

$$x^* := \frac{Mq_0}{-\delta\kappa} + \frac{q_0}{-\zeta'(0)} + nh_0 > 2h_0,\ n \in \mathbb{N},\ \ q(t) := q_0 e^{-\delta t},$$
$$\overline{h}(t) := x^* - \xi(t) + \frac{q(t)}{-\zeta'(0)},\ \ \xi(t) := -\frac{M}{\delta\kappa}q(t).$$

Now we show that $(\overline{U}, \overline{h})$ is an upper solution of the problem (1.1) in $\Sigma := \{(t, x) : 0 \le x \le \overline{h}(t),\ 0 < t < \widetilde{T}\}$. For $1 - \varepsilon_3 \le \zeta \le 1$ we have

$$\overline{U}_t - \overline{U}_{xx} - \overline{U}(1 - \overline{U} - k_1 v) \ge \overline{U}_t - \overline{U}_{xx} - \overline{U}(1 - \overline{U}) \ge q'(t) + \delta q(t) = 0.$$

For the case $-q(t) \le \zeta \le 1 - \varepsilon_3$. When $x^* - \xi(t) \le x \le \overline{h}(t)$, $\zeta(x) = \zeta'(0)x$, we have $\zeta'' = 0$ and $\zeta < 0$. Hence, when $-q(t) \le \zeta \le 1 - \varepsilon_3$, we have

$$\overline{U}_t - \overline{U}_{xx} - \overline{U}(1 - \overline{U}) \ge \xi'\kappa - Mq = 0.$$

Additionally, we deduce from the definitions of M and ν that

$$\overline{h}'(t) \ge \left(\frac{M}{-\kappa} + \frac{\delta}{\zeta'(0)}\right)q_0 \ge \nu q_0 = -\zeta'(0) - \alpha = -\overline{U}_x(t, \overline{h}(t)) - \alpha.$$

The definition of $\overline{U}$ implies $\overline{U}(t, h(t)) = 0$, while the definitions of x^* and q_0 mean $\overline{U}(0, x) \ge u_0(x)$ for $x \in [-h_0, h_0]$. Therefore, by the comparison principle, $u(t, x) \le \overline{U}(t, x)$ in Σ and $h(t) \le \overline{h}(t)$ for $t \in (0, \widetilde{T})$. So $h(t)$ is bounded, thereby $h_\infty < +\infty$ or $h(t)$ converges to 0 within a finite time. If the former holds, from Lemma 4.4, u converges to w_α, this is impossible since there is no such solution when $\alpha \ge \frac{\sqrt{3}}{3}$. Hence $\lim_{t\to+\widetilde{T}} h(t) = 0$ and $\widetilde{T} < +\infty$. Also, from Lemma 4.3, $u(t, x) \to 0$ as $t \to \widetilde{T}$ and $\lim_{t\to+\infty} v(t, x) = \Lambda(x)$. □

5. Sufficient conditions and the proof of Main result 2

5.1. **Sufficient conditions.** In this section, we give some sufficient conditions for spreading or vanishing of $u(t, x)$.

Theorem 5.1. *Let $h_0 > 0$, then the following properties holds:*

(1) choose $\beta < \alpha$ and $u_0(x) < w_\beta(x)$ for $x \in [0, h_0] \subset [0, L_\beta]$, $v_0(x) > \eta_\beta(x)$, where $(w_\beta, \eta_\beta, L_\beta)$ is the solution of (1.10) *with α replaced by β. Then Case (II) in the Main result 2 (see the introduction) happens, that is, u vanishes and v spreads.*

(2) choose $\gamma > \alpha$ and $u_0(x) > w_\gamma(x)$ for $x \in [0, L_\gamma] \subset [0, h_0]$, $v_0(x) < \eta_\gamma(x)$, where (w_γ, η_γ) is the solution of (1.10) *with α replaced by γ. Then Case (I) in the Main result 2 happens, that is, u spreads and v vanishes.*

Proof. (1) Note that $\beta < \alpha$, $u_0(x) < w_\beta(x)$ and $v_0(x) > \eta_\beta(x)$, by Lemma 2.5 and Remark 2.6, we have $u(t, x) \le w_\beta(x)$ for $x \in [0, h(t)]$ and $t > 0$. Thus we have $0 < h_\infty \le L_\beta$ or $\lim_{t\to+\widetilde{T}} h(t) = 0$ for some $\widetilde{T} < +\infty$. If the former case is true, by Lemma 4.4, $u(t, x)$ will converge to $w_\alpha(x)$ and $h(t)$ tends to L_α. This is impossible since we have proved that $h_\infty \le L_\beta$ (note that $L_\beta < L_\alpha$). So the later holds, combining this and Lemma 4.3 we deuce that u vanishes and $\lim_{t\to+\infty} v(t, x) = \Lambda(x)$ locally uniformly in $[0, +\infty)$.

(2) Since $\gamma > \alpha$, $u_0(x) > w_\gamma(x)$ and $v_0(x) < \eta_\gamma(x)$, by Lemma 2.5 and Remark 2.6, we have $u(t, x) \ge w_\gamma(x)$ for $t > 0$ and $x \in [0, h(t)]$, $v(t, x) \le \eta_\gamma(x)$. So $h_\infty < +\infty$ or $h_\infty = +\infty$. The former is also impossible sine $L_\gamma > L_\alpha$ and there is no compactly supported solution satisfying the free boundary

condition. So only $h_\infty = +\infty$ happens. It derives from Lemma 4.2 that $\lim_{t\to+\infty} u(t,x) = \Lambda(x)$ and $\lim_{t\to+\infty} v(t,x) = 0$. □

5.2. **The completion of proof of the Main result 2.** By Lemmas 4.2-4.4, we have Case (I), Case (II) and Case (III). To prove Main result 2, we only need to prove the sharp result (see the following Theorem 5.2).

Theorem 5.2. *Let* (u, v, h) *be the solution of the problem* (1.1) *with* $u_0 = \sigma\phi$ *for some* $\phi \in \mathscr{X}(h_0)$, $\sigma > 0$, *denote* $u(t, x)$ *as* $u(t, x; \sigma\phi)$. *Then there exists*

$$\sigma^* = \sigma^*(h_0, \phi) := \sup\{\sigma : u(t, x; \sigma\phi) \text{ vanishes for } \sigma \in (0, \sigma_0]\} \in (0, +\infty]$$

such that

(1) If $\sigma < \sigma^*$, *Case (II) happens.*
(2) If $\sigma = \sigma^*$, *the transition case happens.*
(3) If $\sigma > \sigma^*$, *Case (I) happens.*

Proof. (1) The case $0 < \sigma < \sigma^*$, by the definition of σ^* and Lemma 2.5, we get that vanishing happens for u. By this and Lemma 4.3, we deduce that Case (II) happens when $0 < \sigma < \sigma^*$.

(2) The case $\sigma = \sigma^*$. In this case, we cannot have Case (II), for otherwise we have, for some large $t_0 > 0$,

$$u(t_0, x) < w_\beta(x), \ \ x \in [0, h(t_0)] \subset [0, L_\beta].$$

and

$$v(t_0, x) \geq \eta_\beta(x) \quad \text{for } x \in [0, +\infty).$$

Due to the continuous dependence of the solution on the initial values, we can find a $\epsilon > 0$ sufficiently small such that the solution $(u_\epsilon, v_\epsilon, h_\epsilon)$ of (1.1) with initial data $(\sigma^* + \epsilon)\phi$ satisfies

$$u_\epsilon(t_0, x) < w_\beta(x), \ \ x \in [0, h(t_0)] \subset [0, L_\beta]$$

and

$$v_\epsilon(t_0, x) \geq \eta_\beta(x) \quad \text{for } x \in [0, +\infty).$$

Hence we can apply Theorem 5.1 that vanishing happens for (u_ϵ, h_ϵ), a contradiction to the definition of σ^*. Thus at $\sigma = \sigma^*$, u cannot vanish. So Case (II) is impossible.

We next prove that spreading of u cannot happen when $\sigma = \sigma^*$. Otherwise, for large $t_0 > 0$, there holds

$$h(t_0) > L_\gamma, \ u(t_0, \cdot) > w_\gamma(\cdot) \ in \ [0, L_\gamma], \quad \text{and} \quad v(t_0, x) < \eta_\gamma(x) \text{ for } x \geq 0, \tag{5.1}$$

where $(w_\gamma, \eta_\gamma, L_\gamma)$ is given in Theorem 5.1 (2). Also, we may choose a small ϵ such that the solution $(u^\epsilon, v^\epsilon, h^\epsilon)$ of (1.10) with $u_0 = (\sigma^* - \epsilon)\phi$ also satisfies (5.1). From Lemma 2.5, we have, for all $t > 0$,

$$u^\epsilon(t_0 + t, x) > w_\gamma(x) \ in \ (0, L_\gamma) \subset [0, h^\epsilon(t)] \quad \text{and } v^\epsilon(t_0 + t, x) < \eta_\gamma(x) \text{ for } x \geq 0.$$

Hence 5.1 (2) implies that spreading happens for (u^ϵ, h^ϵ). But it is a contradiction to the definition of σ^*. Hence when $\sigma = \sigma^*$, there is only the transition case.

(3) The case $\sigma > \sigma^*$. We only need to prove that the transition case cannot happen when $\sigma > \sigma^*$. Let (u_*, v_*, h_*) be the solution of the problem (1.1) with initial data $u_0 = \sigma^*\phi$, and (u_1, v_1, h_1) be the solution of the problem (1.1) with initial data $u_0 = \sigma\phi$ and v_0 is the same one as in the case $\sigma = \sigma^*$. Suppose that the transition case also happens for (u_1, v_1, h_1). Then

$$h_*(t) \to L_\alpha \quad \text{and} \quad h_1(t) \to L_\alpha \text{ as } t \to +\infty. \tag{5.2}$$

By the comparison principle we have, for all $t > 0$,

$$h_*(t) < h_1(t), \ u_*(t, x) < u_1(t, x) \text{ for } x \in (0, h_*(t)]; \quad v_*(t, x) \geq v_1(t, x) \text{ for } x \geq 0.$$

For some $t_0 > 0$, there is small $\epsilon_1 > 0$ such that

$$h_*(t) + \epsilon_1 < h_1(t),\ u_*(t_0, x - \epsilon_1) < u_1(t_0, x)\ in\ [\epsilon_1, h_*(t_0) + \epsilon_1]$$

Thus we have, for all $t > 0$,

$$u_*(t + t_0, x - \epsilon_1) < u_1(t + t_0, x)\ in\ [\epsilon_1, h_*(t + t_0)]$$

and

$$h_*(t + t_0) + \epsilon_1 < h(t + t_0).$$

Combining this and (5.2), letting $t \to +\infty$, we have $L_\alpha + \epsilon \le L_\alpha$. This contradiction implies that the transition case is impossible when $\sigma > \sigma^*$. So only Case (I) happens.

□

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