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SOME GROUP PRESENTATIONS

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Dedicated to H. S. M. Coxeter

1. Background. Let the group $\mathscr{G}$ be presented on three generators a, b, c with three defining relations

(1.1) $a^p b = b a^s,$

(1.2) $b^q c = c b^t,$

(1.3) $c^r a = a c^u,$

where p, q, r, s, t, u are integers, the "parameters" of the presentation. A natural question is whether $\mathscr{G}$ is finite or infinite, and if it is finite, whether it is trivial. The answer is in many cases obvious; for example, if $p = s$, the group is infinite, as is readily seen upon equating b and c with the unit element, which amounts to mapping the group epimorphically on an infinite cyclic group generated by (the image of) a. In some cases the answer is less obvious; for example, if $p = q = r = 1$ and $s = t = u = -1$, then the group is infinite [**3**, § 5], while if $p = q = r = 1$ and $s = t = u = 2$, the group is trivial [**3**, § 23].

Some interest has recently been shown in the case that $p = q = r = 2$ and $s = t = u = 3$, though I am not aware of any published reference to this case (I am indebted to Dr. A. M. Brunner for reminding me of it.). It is constructed in an obvious way from the amalgam of 3 copies of the famous two-generator one-relator non-hopfian group

$$\text{gp}\,(a, b; a^2 b = b a^3)$$

of Baumslag and Solitar [**1**].

In the next three sections, attention will be restricted to parameters that satisfy

(1.4) $2 \leqq p \leqq |s|, \quad 2 \leqq q \leqq |t|, \quad 2 \leqq r \leqq |u|.$

Thus we do not here consider the cases when 0 occurs among the parameters: they tend to be easier and less interesting; nor the cases when 1 occurs among the parameters: they make our method fail. Beyond this we do not restrict the generality by postulating the inequalities (1.4); they can always be satisfied

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by possibly permuting the generators and replacing some of them by their inverses.

2. Normal matrices, and some notation. A word in the generators can be written in the form

(2.1) $$w = a^{\alpha(1)}b^{\beta(1)}c^{\gamma(1)}a^{\alpha(2)} \dots b^{\beta(n)}c^{\gamma(n)}.$$

The case $n = 0$ gives the "empty word" w_0. It is convenient to arrange the exponents in (2.1) in a $3 \times n$ matrix

(2.2) $$m = \begin{bmatrix} \alpha(1) & \alpha(2) \dots \alpha(n) \\ \beta(1) & \beta(2) \dots \beta(n) \\ \gamma(1) & \gamma(2) \dots \gamma(n) \end{bmatrix},$$

with the empty word w_0 giving the empty matrix

(2.20) $$m_0 = \begin{bmatrix} \quad \end{bmatrix}.$$

The words and matrices are in one-to-one correspondence, and while we shall use the language of matrices, everything can easily be translated into words.

We shall use juxtaposition of matrices like juxtaposition of words; thus if we have two matrices

$$m' = \begin{bmatrix} \alpha'(1) \dots \alpha'(n') \\ \beta'(1) \dots \beta'(n') \\ \gamma'(1) \dots \gamma'(n') \end{bmatrix}, \quad m'' = \begin{bmatrix} \alpha''(1) \dots \alpha''(n'') \\ \beta''(1) \dots \beta''(n'') \\ \gamma''(1) \dots \gamma''(n'') \end{bmatrix},$$

then we define the matrix with $n' + n''$ columns

$$m'm'' = \begin{bmatrix} \alpha'(1) \dots \alpha'(n')\, \alpha''(1) \dots \alpha''(n'') \\ \beta'(1) \dots \beta'(n')\, \beta''(1) \dots \beta''(n'') \\ \gamma'(1) \dots \gamma'(n')\, \gamma''(1) \dots \gamma''(n'') \end{bmatrix}.$$

With this "multiplication" the matrices form a semigroup with neutral element m_0.

For each matrix m other than the empty matrix m_0 we define the truncated matrix m^* obtained by omitting the last column. Thus if $m \neq m_0$ is given by (2.2), then

$$m^* = \begin{bmatrix} \alpha(1)\, \alpha(2) \dots \alpha(n-1) \\ \beta(1)\, \beta(2) \dots \beta(n-1) \\ \gamma(1)\, \gamma(2) \dots \gamma(n-1) \end{bmatrix}.$$

Hence

$$m = m^* \begin{bmatrix} \alpha(n) \\ \beta(n) \\ \gamma(n) \end{bmatrix}.$$

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(2.3) *Definition*. The $3 \times n$ matrix of integers given by (2.2) is *normal* if it satisfies the following 8 conditions (2.31)-(2.38).

(2.31) If $\alpha(i) = \beta(i) = 0$, then $i = 1$ and $\gamma(i) \neq 0$;

(2.32) if $\beta(i) = \gamma(i) = 0$, then $i = n$.

In the remaining conditions, n is to be at least 2, and i is to lie in the range $1 \leqq i \leqq n - 1$.

(2.33) If $\beta(i) > 0$ and $\gamma(i) = 0$, then $1 \leqq \alpha(i + 1) \leqq |s| - 1$;

(2.34) if $\beta(i) < 0$ and $\gamma(i) = 0$, then $1 \leqq \alpha(i + 1) \leqq p - 1$;

(2.35) if $\gamma(i) > 0$ and $\alpha(i + 1) = 0$, then $1 \leqq \beta(i + 1) \leqq |t| - 1$;

(2.36) if $\gamma(i) < 0$ and $\alpha(i + 1) = 0$, then $1 \leqq \beta(i + 1) \leqq q - 1$;

(2.37) if $\alpha(i + 1) > 0$, then $0 \leqq \gamma(i) \leqq r - 1$;

(2.38) if $\alpha(i + 1) < 0$, then $1 \leqq \gamma(i) \leqq |u| - 1$.

Note that the empty matrix m_0 satisfies these conditions vacuously; that the null matrix $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ does not satisfy (2.31), but all other single column matrices satisfy the conditions, the last 6 of them vacuously; and that $\gamma(i)$ and $\alpha(i + 1)$ can not simultaneously vanish.

The set of all normal matrices is denoted by M. The following fact is obvious:

(2.4) LEMMA. *If $m = m'm''$ is a normal matrix, then m' and m'' are normal matrices; in particular, if m is a non-empty normal matrix, then m^* is a normal matrix.*

The word w that corresponds to a normal matrix m will be called a *normal word*. The aim is to show that among all the words that represent an element of our group G, there is precisely one normal word, the *normal form* of the element. The procedure is a variant of the well-known method of van der Waerden [**5**]; see also [**3**, §2].

The following abbreviations will be used in the sequel:

(2.5) $p^* = p \operatorname{sign} s$; $q^* = q \operatorname{sign} t$.

(2.6) $\beta^*(m) = |\beta(1)| + |\beta(2)| + \ldots + |\beta(n)|$,

(2.7) $\gamma^*(m) = |\gamma(1)| + |\gamma(2)| + \ldots + |\gamma(n)|$,

where m is given by (2.2). In particular we put, for the empty matrix m_0,

(2.60) $\beta^*(m_0) = 0$,

(2.70) $\gamma^*(m_0) = 0$.

We further put

(2.8) $M_k' = \{m \in M | \beta^*(m) < k\}$,

(2.9) $M_k'' = \{m \in M | \gamma^*(m) < k\}$.

3. The elementary mappings. We now define 6 mappings of the set M of normal matrices into itself; they will be called *elementary* mappings and denoted by

$$\rho(a), \rho(a^{-1}), \rho(b), \rho(b^{-1}), \rho(c), \rho(c^{-1}).$$

We begin with the simplest, namely $\rho(c)$ and $\rho(c^{-1})$; here and later m will always be given by (2.2), and m_0 by (2.20).

(3.10) $m_0\rho(c) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$;

(3.11) if $\gamma(n) \neq -1$, or if $\gamma(n) = -1$ and $\beta(n) \neq 0$ or $\alpha(n) \neq 0$, then

$$m\rho(c) = m^* \begin{bmatrix} \alpha(n) \\ \beta(n) \\ \gamma(n) + 1 \end{bmatrix};$$

(3.12) if $\gamma(n) = -1$ and $\beta(n) = \alpha(n) = 0$, which implies that $n = 1$, then

$$m\rho(c) = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \rho(c) = m_0.$$

(3.20) $m_0\rho(c^{-1}) = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$;

(3.21) if $\gamma(n) \neq 1$, or if $\gamma(n) = 1$ and $\beta(n) \neq 0$ or $\alpha(n) \neq 0$, then

$$m\rho(c^{-1}) = m^* \begin{bmatrix} \alpha(n) \\ \beta(n) \\ \gamma(n) - 1 \end{bmatrix};$$

(3.22) if $\gamma(n) = 1$ and $\beta(n) = \alpha(n) = 0$, which implies that $n = 1$, then

$$m\rho(c^{-1}) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \rho(c^{-1}) = m_0.$$

Then the following facts are easily verified; we omit the proof.

(3.3) LEMMA. *The mappings $\rho(c)$ and $\rho(c^{-1})$ map M to M; they are inverse to each other, and hence are permutations of M.*

The iterates of $\rho(c)$ and $\rho(c^{-1})$ are also defined on M, and for negative exponents γ we interpret

$$\rho(c)^\gamma = \rho(c^{-1})^{-\gamma},$$

and $\rho(c)^0$ as the identity mapping; then all powers $\rho(c)^\gamma$ are defined on M.

Next we define $\rho(b)$ and $\rho(b^{-1})$, by induction on γ^*—see (2.7). First assume $\gamma^*(m) = 0$, so that (see (2.9))

$$m = \begin{bmatrix} \alpha(1) \dots \alpha(n) \\ \beta(1) \dots \beta(n) \\ 0 \quad \dots \quad 0 \end{bmatrix} \in M_1''.$$

We define

(3.40) $$m_0\rho(b) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix};$$

(3.41) if $\beta(n) \neq -1$, or if $\beta(n) = -1$ and $\alpha(n) \neq 0$, then

$$m\rho(b) = m^* \begin{bmatrix} \alpha(n) \\ \beta(n) + 1 \\ 0 \end{bmatrix};$$

(3.42) if $\beta(n) = -1$ and $\alpha(n) = 0$, which implies that $n = 1$, then

$$m\rho(b) = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \rho(b) = m_0.$$

Similarly we define

(3.50) $$m_0\rho(b^{-1}) = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix};$$

(3.51) if $\beta(n) \neq 1$, or if $\beta(n) = 1$ and $\alpha(n) \neq 0$, then

$$m\rho(b^{-1}) = m^* \begin{bmatrix} \alpha(n) \\ \beta(n) - 1 \\ 0 \end{bmatrix};$$

(3.52) if $\beta(n) = 1$ and $\alpha(n) = 0$, which implies that $n = 1$, then

$$m\rho(b^{-1}) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \rho(b^{-1}) = m_0.$$

These definitions start the induction. Now one verifies without difficulty that so far, that is for M_1'', (3.60) *the mappings* $\rho(b)$ *and* $\rho(b^{-1})$ *are inverse to each other and leave* γ^* *invariant.*

It follows that the iterates of $\rho(b)$ and $\rho(b^{-1})$ are also defined on M_1'', and if, for negative exponents β, we interpret

$$m\rho(b)^{\beta} = \rho(b^{-1})^{-\beta},$$

and $\rho(b)^0$ as the identity mapping, then all powers $\rho(b)^{\beta}$ are defined on M_1''.

To continue the induction, we assume that all powers $\rho(b)^{\beta} = \rho(b^{-1})^{-\beta}$ are defined on M_k'' and there satisfy (3.60). To extend the definition to M_{k+1}'', we take a matrix $m \in M$ with $\gamma^*(m) = k$, and define:

(3.43) If $\gamma(n) \neq 0$, then

$$m\rho(b) = m\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix};$$

(3.44) if $\gamma(n) = 0$ and $\alpha(n) \neq 0$, or if $\gamma(n) = \alpha(n) = 0$ and $\gamma(n-1) > 0$ and $\beta(n) \leqq |t| - 2$, or if $\gamma(n) = \alpha(n) = 0$ and $\gamma(n-1) < 0$ and $\beta(n) \leqq q - 2$, we put

$$m\rho(b) = m^*\begin{bmatrix} \alpha(n) \\ \beta(n) + 1 \\ 0 \end{bmatrix};$$

(3.45) if $\gamma(n) = \alpha(n) = 0$ and $\gamma(n-1) > 0$ and $\beta(n) = |t| - 1$, we put

$$m\rho(b) = m^*\rho(c^{-1})\rho(b)^{q^*}\rho(c).$$

Here q^* is defined by (2.5), and we note that

$$\gamma^*(m^*\rho(c^{-1})) = \gamma^*(m) - 1 = k - 1,$$

so that the action of $\rho(b)^{q^*}$ on $m^*\rho(c^{-1})$ is defined.

(3.46) If $\gamma(n) = \alpha(n) = 0$ and $\gamma(n-1) < 0$ and $\beta(n) = q - 1$, we put

$$m\rho(b) = m^*\rho(c)\rho(b)^{t}\rho(c^{-1}).$$

The justification is as in (3.45). Similarly we define:

(3.53) If $\gamma(n) \neq 0$, then

$$m\rho(b^{-1}) = m\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix};$$

(3.54) if $\gamma(n) = 0$ and $\alpha(n) \neq 0$, or if $\gamma(n) = \alpha(n) = 0$ and $\beta(n) \geqq 2$, we put

$$m\rho(b^{-1}) = m^*\begin{bmatrix} \alpha(n) \\ \beta(n) - 1 \\ 0 \end{bmatrix};$$

(3.55) if $\gamma(n) = \alpha(n) = 0$ and $\beta(n) = 1$, we put $m\rho(b^{-1}) = m^*$.

It is not immediately obvious, but not difficult to check, that these cases exhaust all possibilities permitted by (2.31)—(2.38), and that the induction hypothesis (3.60) remains satisfied. Thus we have the following lemma:

(3.61) LEMMA. *The mappings* $\rho(b)$ *and* $\rho(b^{-1})$ *map* M *to* M; *they are inverse to each other, and thus are permutations of* M; *moreover they leave* γ^* *invariant.*

We omit the detailed verification.

Finally we define $\rho(a)$ and $\rho(a^{-1})$, by induction on β^*—see (2.6). First assume $\beta^*(m) = 0$, so that (see (2.8))

$$m = \begin{bmatrix} \alpha(1) \ldots \alpha(n) \\ 0 \quad \ldots \quad 0 \\ \gamma(1) \ldots \gamma(n) \end{bmatrix} \in M_1'.$$

We put

$$(3.70) \quad m_0\rho(a) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Next we write

$$\gamma(n) = vr + x \quad \text{with } 0 \leqq x < r,$$

and define

(3.71) if $x \neq 0$, then

$$m\rho(a) = m^* \begin{bmatrix} \alpha(n) & 1 \\ 0 & 0 \\ x & vu \end{bmatrix};$$

(3.72) if $x = 0$ and $v \neq 0$, or if $x = v = 0$, which means $\gamma(n) = 0$ and implies $n = 1$, and if $\alpha(n) \neq -1$, then

$$m\rho(a) = m^* \begin{bmatrix} \alpha(n) + 1 \\ 0 \\ vu \end{bmatrix};$$

(3.73) if $x = v = 0$, so that $n = 1$, and if $\alpha(1) = -1$, then

$$m\rho(a) = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \rho(a) = m_0.$$

Similarly we put

$$(3.80) \quad m_0\rho(a^{-1}) = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}.$$

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Next we write

$$\gamma(n) = yu + z \quad \text{with } 0 \leqq z < |u|,$$

and define

(3.81) if $z \neq 0$, then

$$m\rho(a^{-1}) = m^* \begin{bmatrix} \alpha(n) & -1 \\ 0 & 0 \\ z & yr \end{bmatrix};$$

(3.82) if $z = 0$ and $y \neq 0$, or if $z = y = 0$, which means $\gamma(n) = 0$ and implies $n = 1$, and if $\alpha(n) \neq 1$, then

$$m\rho(a^{-1}) = m^* \begin{bmatrix} \alpha(n) - 1 \\ 0 \\ yr \end{bmatrix};$$

(3.83) if $z = y = 0$, so that $n = 1$, and if $\alpha(1) = 1$, then

$$m\rho(a^{-1}) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \rho(a^{-1}) = m_0.$$

These definitions start the induction. Now one verifies without difficulty that so far, that is for M_1',

(3.90) *the mappings $\rho(a)$ and $\rho(a^{-1})$ are inverse to each other and leave β^* invariant.*

It follows that the iterates of $\rho(a)$ and $\rho(a^{-1})$ are also defined on M_1', and if, for negative exponents α, we interpret

$$\rho(a)^\alpha = \rho(a^{-1})^{-\alpha},$$

and $\rho(a)^0$ as the identity mapping, then all powers $\rho(a)^\alpha$ are defined on M_1'. To continue the induction, we assume that all powers $\rho(a)^\alpha = \rho(a^{-1})^{-\alpha}$ are defined on M_k' and there satisfy (3.90). To extend the definition to M_{k+1}', we take a matrix $m \in M$ with $\beta^*(m) = k$ and define, still with

$$\gamma(n) = vr + x = yu + z \quad \text{with } 0 \leqq x < r, 0 \leqq z < |u|:$$

(3.74) if $x \neq 0$, or if $x = 0$ and $\beta(n) \neq 0$, then

$$m\rho(a) = m^* \begin{bmatrix} \alpha(n) & 1 \\ \beta(n) & 0 \\ x & vu \end{bmatrix};$$

(3.75) if $x = \beta(n) = 0$ and $\gamma(n-1) \neq 0$, or if $x = \beta(n) = \gamma(n-1) = 0$ and $\beta(n-1) > 0$ and $\alpha(n) \leqq |s| - 2$, or if $x = \beta(n) = \gamma(n-1) = 0$ and $\beta(n-1) < 0$ and $\alpha(n) \leqq p - 2$,

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then

$$m\rho(a) = m^* \begin{bmatrix} \alpha(n) + 1 \\ 0 \\ vu \end{bmatrix};$$

(3.76) if $x = \beta(n) = \gamma(n-1) = 0$ and $\beta(n-1) > 0$ and $\alpha(n) = |s| - 1$, then
$m\rho(a) = m^*\rho(b^{-1})\rho(a)^{p^*}\rho(b)\rho(c)^{vu}$.

Here p^* is defined by (2.5), and we note that

$$\beta^*(m^*\rho(b^{-1})) = \beta^*(m) - 1 = k - 1,$$

so that the action of $\rho(a)^{p^*}$ on $m^*\rho(b^{-1})$ is defined.

(3.77) If $x = \beta(n) = \gamma(n-1) = 0$ and $\beta(n-1) < 0$ and $\alpha(n) = p - 1$, then
$m\rho(a) = m^*\rho(b^{-1})\rho(a)^s\rho(b)\rho(c)^{vu}$.

The justification of this is as in (3.76). Similarly we define:

(3.84) If $z \neq 0$, or if $z = 0$ and $\beta(n) \neq 0$, then

$$m\rho(a^{-1}) = m^* \begin{bmatrix} \alpha(n) & -1 \\ \beta(n) & 0 \\ z & yr \end{bmatrix};$$

(3.85) if $z = \beta(n) = 0$ and $\alpha(n) \geqq 2$, then

$$m\rho(a^{-1}) = m^* \begin{bmatrix} \alpha(n) - 1 \\ 0 \\ yr \end{bmatrix};$$

(3.86) if $z = \beta(n) = 0$ and $\alpha(n) = 1$, then

$$m\rho(a^{-1}) = m^*\rho(c)^{yr}.$$

Again it is not immediately obvious, but not difficult to check, that these cases exhaust all possibilities permitted by (2.31)—(2.38), and that the induction hypothesis (3.90) remains satisfied. Thus we have the following lemma.

(3.91) LEMMA. *The mappings $\rho(a)$ and $\rho(a^{-1})$ map M to M; they are inverse to each other, and thus are permutations of M; moreover they leave β^* invariant.*

We omit the detailed verification.

4. Discussion of the group. The elementary mappings $\rho(a)$, $\rho(b)$, $\rho(c)$ have been shown to be permutations of the set M of normal matrices, with inverses

$$\rho(a)^{-1} = \rho(a^{-1}), \quad \rho(b)^{-1} = \rho(b^{-1}), \quad \rho(c)^{-1} = \rho(c^{-1});$$

they therefore generate a group, say $\mathscr{P}$, of permutations of M. If the matrix

$$m = \begin{bmatrix} \alpha(1) \ldots \alpha(n) \\ \beta(1) \ldots \beta(n) \\ \gamma(1) \ldots \gamma(n) \end{bmatrix}$$

is normal, then one readily verifies that

$$m = m_0\rho(a)^{\alpha(1)}\rho(b)^{\beta(1)}\rho(c)^{\gamma(1)} \ldots \rho(a)^{\alpha(n)}\rho(b)^{\beta(n)}\rho(c)^{\gamma(n)}.$$

It follows that $\mathscr{P}$ acts transitively on M.

Let g be an element of our given group $\mathscr{G}$. Then g can be represented by a word in the generators a, b, c. The corresponding word in $\rho(a)$, $\rho(b)$, $\rho(c)$ is an element of $\mathscr{P}$ that we denote by $\rho(g)$. It is not obvious that this notation is legitimate, that is to say that the permutation $\rho(g)$ depends only on the element g of $\mathscr{G}$, not on the particular word in a, b, c chosen to represent g. To prove that this is the case, one needs to show that $\rho(g)$ is the identity permutation ι of W if, and only if, g is the unit element of $\mathscr{G}$. This is done by, firstly, verifying that $\rho(a)$, $\rho(b)$, $\rho(c)$ satisfy the relations

(4.1) $\rho(a)^p\rho(b) = \rho(b)\rho(a)^s$,

(4.2) $\rho(b)^q\rho(c) = \rho(c)\rho(b)^t$,

(4.3) $\rho(c)^r\rho(a) = \rho(a)\rho(c)^u$

that correspond to the defining relations (1.1)—(1.3) of $\mathscr{G}$; and secondly, that they satisfy no relations that do not follow from these. The first part of the verification is straightforward, though laborious; the second part, which amounts to showing the uniqueness of the normal form, follows the argument in [**3**, § 2]. We omit the verifications. The argument will show that

$$\rho : \mathscr{G} \to \mathscr{P} \quad \text{defined by } g \to \rho(g)$$

is an isomorphism. As $\mathscr{P}$ acts transitively on the evidently infinite set M, it follows that $\mathscr{G}$ is infinite.

Moreover one can use the same argument to show that the subgroup of $\mathscr{G}$ generated by a, b is defined by the relation (1.1); and analogously for the subgroups generated by b, c and by c, a, respectively. To sum up, we have the following facts:

(4.4) THEOREM. *The group* $\mathscr{G}$ *generated by* a, b, c *with defining relations* (1.1), (1.2), (1.3) *subject to the inequalities* (1.4) *is infinite. It is the generalized free product of the subgroups generated by* a, b *and defined by* (1.1), *by* b, c *with* (1.2), *and by* c, a *with* (1.3).

It is clear how the normal form of an element of $\mathscr{G}$, given as some word in the generators a, b, c, can be computed. As a consequence we have the following fact:

(4.5) COROLLARY. *The word problem for the presentation* (1.1)—(1.3) *of* $\mathcal{G}$ *is solvable.*

5. Further results. It is clear that what has here been done for group presentations with 3 generators can be extended to analogous group presentations with more than 3 generators. However, for 4 or more generators the method of [**3**, § 23] is available, and it is unnecessary to resort to the complicated normal form argument of the present paper.

We turn to the question which of our groups $\mathcal{G}$, defined by (1.1)—(1.3), have non-trivial finite epimorphic images, or, equivalently, proper subgroups of finite index. This is answered by the following theorem.

(5.1) THEOREM. *The group* $\mathcal{G}$ *defined by* (1.1)—(1.3) *has non-trivial finite factor groups if, and only if, its parameters satisfy the inequality*

(5.11) $|(s-p)(t-q)(u-r)| \neq 1.$

Proof. If the inequality (5.11) is satisfied, then at least one of the differences $|s-p|$, $|t-q|$, $|u-r|$ is different from 1. Let $s - p = d > 1$; then G can be mapped epimorphically on a cyclic group of order d by mapping a on one of its generators and b and c on the unit element; and the same is true if $p - s = d > 1$. If $p = s$, we similarly map G epimorphically on an infinite cyclic group, and this on an arbitrary non-trivial finite cyclic group. Conversely assume that (5.11) is not satisfied. No generality is lost by assuming that

$$s = p + 1, \quad t = q + 1, \quad u = r + 1.$$

If one of the parameters is zero. the corresponding generator equals the unit element, and so then do the others. Thus we may further assume that

$$1 \leqq p, \quad 1 \leqq q, \quad 1 \leqq r.$$

Let, in some finite epimorphic image of G, the images of a, b, c have orders α, β, γ, respectively, and assume that, if possible, $1 < \alpha$, $1 < \beta$, $1 < \gamma$. Now from the relation

$$a^p b = b a^{p+1}$$

we get

$$a^{p^\beta} = a^{(p+1)^\beta},$$

whence

$$\alpha | (p+1)^\beta - p^\beta;$$

and similarly

$$\beta | (q+1)^\gamma - q^\gamma,$$

$$\gamma | (r+1)^\alpha - r^\alpha.$$

Denote the least prime divisor of α by π_α, that of β by π_β, that of γ by π_γ. Now

$$(p+1)^\beta \equiv p^\beta \pmod{\pi_\alpha}$$

implies that

$$\pi_\alpha - 1|\beta,$$

and hence that

$$\pi_\beta < \pi_\alpha.$$

Similarly

$$\pi_\alpha < \pi_\gamma < \pi_\beta,$$

and finally

$$\pi_\alpha < \pi_\alpha.$$

This is impossible. Hence one of α, β, γ equals 1; but then they all do, and the epimorphic image of $\mathscr{G}$ is trivial. This completes the proof of the theorem. The argument is that of Higman [**2**] (see also [**3**, § 23]).

(5.2) COROLLARY. *The group generated by a, b, c with defining relations*

$$a^pb = ba^{p+1},$$
$$b^qc = cb^{q+1},$$
$$c^ra = ac^{r+1},$$
$$a^\alpha = 1,$$

with $\alpha \neq 0$, *is trivial.*

This is a corollary of the proof rather than of the theorem.

A recent result of Post [**4**] deals with many of the cases for which our method fails. He has shown that if, in our notation,

$$1 = r < u, \quad 0 < p < s, \quad 0 < q < t,$$

and

$$(p, s) = (q, t) = 1,$$

then G is finite; and if, moreover, $u = 2$, $s = p + 1$, and $r = q + 1$, then G is trivial. (I am grateful to Dr. Michael J. Post for permitting me to quote his unpublished paper.)

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