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TWO MULTIPLIER THEOREMS FOR $H^1(U^2)$

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1. Introduction

Let $H^1(U^2)$ be the Hardy space of the bidisc as described in (3). Each function $f \in H^1(U^2)$ has a Taylor expansion of the form $f(z, w) = \Sigma_{n,m\geq 0}\hat{f}(n, m)z^n w^m$. For $0 < p < \infty$, a doubly-indexed sequence $(\lambda_{nm})_{n,m\geq 0}$ is said to be a multiplier of $H^1(U^2)$ into l^p if

$$\sum_{n,m\geq 0} |\hat{f}(n, m)\lambda_{nm}|^p < \infty \quad \text{for each } f \in H^1(U^2).$$

This paper is concerned with the cases $p = 2$ and $p = 1$. Theorem 1 characterises the multipliers of $H^1(U^2)$ into l^2 and is an analogue in two variables of an old result of Hardy and Littlewood. Theorem 2 characterises the sequences $(a_n)_{n\geq 0}$ such that $(a_{n+m})_{n,m\geq 0}$ is a multiplier of $H^1(U^2)$ into l^1. For the special class of multipliers which it describes, Theorem 2 goes substantially beyond the well known but ineffectual characterisation of the multipliers of $H^1(U)$ into l^1. (The one-dimensional results mentioned are given as Theorems 6.7 and 6.8 in (1). Their proofs depend on the well known factorisation properties of functions in $H^1(U)$, and so two-dimensional theorems can not be established by a mere repetition of the one-dimensional proofs.)

We mention that versions of our theorems can be formulated for the spaces $H^1(U^n)$ ($n = 3, 4, \ldots$), but for notational reasons we have contented ourselves with $H^1(U^2)$.

2. The theorems

We begin by establishing some notation. Let T be the unit circle in the complex plane, let m_1 be normalised Lebesgue measure on T, and let m_2 be the associated product measure on T^2. Fix $f \in H^1(U^2)$. It is well known that for (m_2-) almost every $(z, w) \in T^2$ the limit $\lim_{r\to 1^-} f(rz, rw)$ exists. If we write $f(z, w)$ for this limit when it exists, then $f(z, w) \in L^1(T^2)$ $(= L^1(T^2, m_2))$. In fact, the set of such $f(z, w)$ so obtained is a closed subspace of $L^1(T^2)$, and so $H^1(U^2)$ is a Banach space under the norm

$$f \to \int_{T^2} |f(z, w)|\, dm_2(z, w) = \|f\|.$$

(Our notation identifies a function $f(z, w)$ in $H^1(U^2)$ with its boundary function on T^2.

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We will also follow this convention in the case of functions in $H^1(U)$ and their boundary functions on T.)

To state our theorems we will need the following terminology. Let $I_{-1}=\emptyset$, $I_0=\{0\}$, and $I_k=\{2^{k-1},2^{k-1}+1,\ldots,2^k-1\}$ for $k=1,2,\ldots$. For $j,k\geq 0$, let $I_{jk}=I_j\times I_k$.

Theorem 1. *For a doubly-indexed sequence* $\lambda=(\lambda_{nm})_{n,m\geq 0}$, *the following are equivalent*:

a) $\sup_{j,k\geq 0}\left(\sum_{(n,m)\in I_{jk}}|\lambda_{nm}|^2\right)<\infty$;

b) λ *is a multiplier of* $H^1(U^2)$ *into* l^2;

c) $\sum_{n=0}^{N}\sum_{m=0}^{M} n^2m^2|\lambda_{nm}|^2=O(N^2M^2)$.

Proof. a)→b) It suffices to show that if $f\in H^1(U^2)$, then

$$\sum_{j,k\geq 0}\sup_{(n,m)\in I_{jk}}|\hat{f}(n,m)|^2<\infty.$$

We will do this by exhibiting polynomials $p_{jk}=\sum_{n,m\geq 0}\hat{p}_{jk}(n,m)z^nw^m$ such that

i) $\hat{p}_{jk}(n,m)=\hat{f}(n,m)$ if $(n,m)\in I_{jk}$, and

ii) $\sum_{j,k\geq 0}\|p_{jk}\|^2<\infty$.

(Recall that $|\hat{h}(n,m)|\leq\|h\|$ for any $h\in H^1(U^2)$ and any $n,m\geq 0$.)

Theorem 5 of (4) implies the existence of a constant C and of sequences of numbers $\{c_{jn}\}_{n\in I_{j-1}}$ $(j=1,2,\ldots)$ and $\{d_{jn}\}_{n\in I_{j+1}}$ $(j=0,1,2,\ldots)$ such that the following holds: if, for $g(z)=\sum_{n\geq 0}\hat{g}(n)z^n\in H^1(U)$, we define

$$S_jg(z)=\sum_{n\in I_{j-1}}c_{jn}\hat{g}(n)z^n$$

$$\Delta_jg(z)=\sum_{n\in I_j}\hat{g}(n)z^n$$

$$T_jg(z)=\sum_{n\in I_{j+1}}d_{jn}\hat{g}(n)z^n,$$

and if

$$\tilde{\Delta}_jg(z)=S_jg(z)+\Delta_jg(z)+T_jg(z),$$

then

$$\int_T\left(\sum_{j\geq 0}|\tilde{\Delta}_jg(z)|^2\right)^{1/2}dm_1(z)\leq C\int_T|g(z)|\,dm_1(z).$$

An application of Minkowski's inequality (see, for example, p. 271 of (5)) thus yields

$$\left(\sum_{j\geq 0}\left[\int_T|\tilde{\Delta}_jg(z)|\,dm_1(z)\right]^2\right)^{1/2}\leq C\int_T|g(z)|\,dm_1(z). \tag{1}$$

We will establish the existence of polynomials p_{jk} satisfying i) and ii) by iterating (1).

Fix $f\in H^1(U^2)$ and write $f_w(z)=f(z,w)$. Using (1), Minkowski's inequality, and Fubini's theorem we get

$$C\|f\| = C\int_T\int_T |f_w(z)|\,dm_1(z)\,dm_1(w)$$
$$\geq \int_T\Big(\sum_{j\geq 0}\Big[\int_T |\tilde{\Delta}_j f_w(z)|\,dm_1(z)\Big]^2\Big)^{1/2} dm_1(w)$$
$$\geq \Big(\sum_{j\geq 0}\Big[\int_T\int_T |\tilde{\Delta}_j f_w(z)|\,dm_1(w)\,dm_1(z)\Big]^2\Big)^{1/2}. \tag{2}$$

Now, writing $f_{jz}(w) = \tilde{\Delta}_j f_w(z)$, we obtain from (1) that

$$C\int_T |f_{jz}(w)|\,dm_1(w) \geq \Big(\sum_{k\geq 0}\Big[\int_T |\tilde{\Delta}_k f_{jz}(w)|\,dm_1(w)\Big]^2\Big)^{1/2}.$$

Another application of Minkowski's inequality gives

$$C\int_T\int_T |f_{jz}(w)|\,dm_1(w)\,dm_1(z) \geq \Big(\sum_{k\geq 0}\|\tilde{\Delta}_k f_{jz}(w)\|^2\Big)^{1/2}.$$

Combining this with (2), we have

$$C^2\|f\| \geq \Big(\sum_{j,k\geq 0}\|\tilde{\Delta}_k f_{jz}(w)\|^2\Big)^{1/2},$$

and so it suffices to take $p_{jk}(z, w) = \tilde{\Delta}_k f_{jz}(w)$. (It is easy to verify that $\hat{p}_{jk}(n, m) = \hat{f}(n, m)$ if $(n, m) \in I_{jk}$.)

b)→c) The argument is analogous to the proof of the necessity of the condition given in Theorem 6.6 of (1). (Choose $f(z, w) = [(1 - rz)(1 - rw)]^{-2}$, $0 < r < 1$.)

c)→a) The proof is very easy and so is omitted.

As a corollary we state a two-dimensional version of a theorem of Paley and Rudin.

Corollary. *For a set E of ordered pairs of nonnegative integers, the following are equivalent*:

a) $\sup_{j,k\geq 0} \operatorname{card}(E \cap I_{jk}) < \infty$;

b) $\sum_{(n,m)\in E} |\hat{f}(n, m)|^2 < \infty$ *for each* $f \in H^1(U^2)$.

Theorem 2. *For a sequence* $(a_n)_{n\geq 0}$ *of numbers, the following are equivalent*:

a) $\sum_{n\in I_j} |a_n| = 0(2^{-j})$;

b) *the doubly-indexed sequence* $(a_{n+m})_{n,m\geq 0}$ *is a multiplier of* $H^1(U^2)$ *into* l^1;

c) *there exists a function* $h \in L^\infty(T^2)$ *satisfying* $\hat{h}(n, m) = \int_{T^2} h(z, w)\bar{z}^n\bar{w}^m\,dm_2(z, w) \geq |a_{n+m}|$ *for* $n, m \geq 0$;

d) $\sum_{n=1}^{N} n^2|a_n| = O(N)$.

Proof. a)→b) It suffices to show that if $f \in H^1(U^2)$, then

$$\sum_{j\geq 0} 2^{-j} \sup_{n\in I_j} \sum_{l=0}^{n} |f(l, n-l)| < \infty. \tag{3}$$

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To establish (3) we will again use inequality (1). It follows from this inequality that there exists a constant C such that

$$\left(\sum_{j\geq 0}\sup_{n\in I_j}|\hat{g}(n)|^2\right)^{1/2}\leq C\int_T|g(z)|\,dm_1(z),\ g\in H^1(U). \tag{4}$$

Now fix a polynomial $f\in H^1(U^2)$ and, for $j=0,1,\ldots$, let $n_j\in I_j$ be such that

$$\sum_{l=0}^{n_j}|\hat{f}(l,n_j-l)|=\sup_{n\in I_j}\sum_{l=0}^{n}|\hat{f}(l,n-l)|. \tag{5}$$

Applying (4) to the homogeneous expansion of f,

$$f(\zeta z,\zeta w)=\sum_{n\geq 0}\zeta^n f_n(z,w)\ \left(f_n(z,w)=\sum_{l=0}^{n}\hat{f}(l,n-l)z^lw^{n-l}\right),$$

we have

$$\left(\sum_{j\geq 0}|f_{n_j}(z,w)|^2\right)^{1/2}\leq C\int_T|f(\zeta z,\zeta w)|\,dm_1(\zeta),$$

and so

$$\int_{T^2}\left(\sum_{j\geq 0}|f_{n_j}(z,w)|^2\right)^{1/2}dm_2(z,w)\leq C\int_{T^2}\int_T|f(\zeta z,\zeta w)|\,dm_1(\zeta)\,dm_2(z,w)=C\|f\|. \tag{6}$$

On the other hand,

$$\begin{aligned}\int_{T^2}\left(\sum_{j\geq 0}|f_{n_j}(z,w)|^2\right)^{1/2}dm_2(z,w)&=\int_{T^2}\left(\sum_{j\geq 0}\left|\sum_{l=0}^{n_j}f(l,n_j-l)z^lw^{n_j-l}\right|^2\right)^{1/2}dm_2(z,w)\\&=\int_T\int_T\left(\sum_{j\geq 0}\left|\sum_{l=0}^{n_j}\hat{f}(l,n_j-l)(z\bar{w})^l\right|^2\right)^{1/2}dm_1(z)\,dm_1(w)\\&=\int_T\left(\sum_{j\geq 0}\left|\sum_{l=0}^{n_j}\hat{f}(l,n_j-l)z^l\right|^2\right)^{1/2}dm_1(z).\end{aligned}$$

Thus it follows from (6) that

$$\int_T\left(\sum_{j\geq 0}\left|\sum_{l=0}^{n_j}\hat{f}(l,n_j-l)z^{2^j-1+l}\right|^2\right)^{1/2}dm_1(z)\leq C\|f\|.$$

Now let $r_j(t)$ be the jth Rademacher function ($j=0,1,2,\ldots$). The inequality above implies that

$$\int_T\int_0^1\left|\sum_{j\geq 0}r_j(t)\sum_{l=0}^{n_j}\hat{f}(l,n_j-l)z^{2^j-1+l}\right|dt\,dm_1(z)\leq C\|f\|.$$

With Fubini's theorem this shows that there exists a sequence $\epsilon_0,\epsilon_1,\epsilon_2,\ldots$ with each $\epsilon_j=\pm 1$ such that

$$\int_T\left|\sum_{j\geq 0}\epsilon_j\sum_{l=0}^{n_j}\hat{f}(l,n_j-l)z^{2^j-1+l}\right|dm_1(z)\leq C\|f\|.$$

Combining this with Hardy's inequality, which states that

$$\sum_{n=0}^{\infty}|\hat{g}(n)|/(n+1)\leq\pi\int_T|g(z)|\,dm_1(z),\quad g\in H^1(U),$$

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we have

$$\sum_{j\geq 0}\sum_{l=0}^{n_j} |\hat{f}(l, n_j - l)|/(2^j + l) \leq \pi C\|f\|.$$

Since $n_j \leq 2^j - 1$, we find that

$$\sum_{j\geq 0} 2^{-j}\sum_{l=0}^{n_j} |\hat{f}(l, n_j - l)| \leq 2\pi C\|f\|$$

and so, by (5),

$$\sum_{j\geq 0} 2^{-j}\sup_{n\in I_j}\sum_{l=0}^{n} |\hat{f}(l, n - l)| \leq 2\pi C\|f\|$$

for polynomial $f \in H^1(U^2)$. This implies (3) for all $f \in H^1(U^2)$.

b)→c) If $(a_{n+m})_{n,m\geq 0}$ is a multiplier of $H^1(U^2)$ into l^1, then $f \mapsto \Sigma_{n,m\geq 0}\hat{f}(n, m)|a_{n+m}|$ defines a continuous linear functional on $H^1(U^2)$. Thus it follows from the Hahn-Banach theorem that there exists $h \in L^\infty(T^2)$ with $\hat{h}(n, m) = |a_{n+m}|$ if $n, m \geq 0$.

c)→d) For $0 < r < 1$ let $f_r(z, w) = [(1 - rz)(1 - rw)]^{-2}$. Then

$$\hat{f}_r(n, m) = (n + 1)(m + 1)r^{n+m}$$

and $\|f_r\| = O([1 - r]^{-2})$

Thus

$$\begin{aligned}\sum_{n\geq 0} |a_n| r^n \sum_{l=0}^{n} (l + 1)(n - l + 1) &= \sum_{p,q\geq 0} |a_{p+q}|\hat{f}_r(p, q)\\ &\leq \sum_{p,q\geq 0} \hat{h}(p, q)\hat{f}_r(p, q) = \int_{T^2} h(z, w)\overline{f_r(z, w)}\, dm_2(z, w)\\ &\leq \|h\|_{L^\infty(T^2)}\|f_r\| = O([1 - r]^{-2}).\end{aligned}$$

Choosing $r = 1 - (1/N)$ this gives

$$\sum_{n=0}^{N} n^3|a_n| = O(N^2),$$

which is equivalent to d).

d)→a) We omit the very easy proof.

Finally we remark that, by Corollary 15 of (2), the equivalence of b) and c) with a_{n+m} replaced by λ_{nm} is equivalent to a positive answer to the question in item (b), p. 68 of (3).

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