17

Basic elements

Quantum Electrodynamics (QED) is the most accurate physical theory we have. Its development in the late 1940's by Dyson, Feynman, Schwinger, Tomonaga and others is one of man's great intellectual triumphs. The key original papers are collected in a volume edited by Schwinger [Sc58]; particularly influential are [Fe49, Fe49a, Dy49]. QED is the culmination of the development of electrodynamics, special relativity, and quantum mechanics. The understanding of QED, in terms of covariance, local gauge invariance, renormalization, and Feynman diagrams laid the basis for all modern relativistic quantum field theories of the fundamental interactions. Since electron scattering involves the electromagnetic interaction of relativistic (massless) Dirac particles, QED plays a central role in the analysis.

The content of QED can be expressed in terms of a set of *Feynman diagrams* with corresponding *Feynman rules* for the S-matrix. We will not derive these here, as that takes us too far afield; their derivation can be found in any standard text [Bj65, Fe71], or course (e.g. [Wa91]). The components of the diagrams are shown in Fig. 17.1. The rules, in the conventions used in this book, are as follows:

1. Draw all topologically distinct connected diagrams;
2. Include a factor of $(-i)(-ie) = -e$ for each order of perturbation theory. Here e is algebraic, and for an electron $e = -|e|$;
3. Include a factor of γ_μ for each vertex [Fig. 17.1(a)];
4. Include a factor of

$$\frac{-i}{(2\pi)^4}\frac{1}{i\gamma_\mu p_\mu + m_e} \tag{17.1}$$

for each fermion (i.e. electron) propagator [Fig. 17.1(b)];

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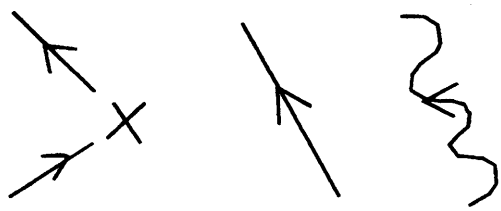

Fig. 17.1. Basic elements of Feynman rules for the S-matrix in QED: (a) vertex; (b) fermion propagator; (c) effective photon propagator.

5. Include a factor of

$$\frac{-i}{(2\pi)^4}\frac{\delta_{\mu\nu}}{q^2} \tag{17.2}$$

for each "effective" photon propagator [Fig. 17.1(c)];[1]

6. Include a wave function for each of the external particles, e.g.

$$\frac{1}{\sqrt{\Omega}}u(p) \text{ incoming fermion;} \quad \frac{1}{\sqrt{2\omega\Omega}}\varepsilon_\mu^{(\lambda)} \text{ incoming photon} \tag{17.3}$$

For a photon with polarization λ in the Coulomb gauge, $\varepsilon_\mu^{(\lambda)} = (\boldsymbol{\varepsilon}^{(\lambda)}, 0)$ and $\boldsymbol{\varepsilon}^{(\lambda)} \cdot \mathbf{k} = 0$;

7. Read along fermion lines;
8. Include a factor of $(2\pi)^4\delta^{(4)}(\Delta p)$ at each vertex;
9. Integrate over all internal momenta $\int d^4q \equiv \int d^3q\, dq_0$;
10. Include a factor of (-1) for each closed fermion loop.

Here we simply treat the hadronic target as an external field, bringing an electromagnetic interaction into the electron line, which we represent by a wavy line ending in a cross. For this component:

11. Include a factor for the external field

$$\frac{a_\mu(q)}{(2\pi)^4} \tag{17.4}$$

[1] This result can be obtained by starting in the Coulomb gauge and then combining the terms coming from the Coulomb interaction (each interaction of order e^2) with those coming from transverse photon exchange (each of order e) in the S-matrix. Terms in q_μ or q_ν in the photon propagator do not contribute to the S-matrix because of current conservation [Bj65, Wa91].

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where the external vector potential has the four-dimensional Fourier transform

$$A_\mu^{\rm ext}(x) = \int e^{iq\cdot x}\frac{a_\mu(q)}{(2\pi)^4}d^4q \tag{17.5}$$

We shall be content here to work to first order in the external field.[2]

Since the electron is light, it can easily radiate as it accelerates, which it does when scattering from a hadronic target. In computing the lowest order radiative corrections to the process of electron scattering, one can consistently confine the analysis to the electron line since it carries charge and runs completely through a diagram from beginning to end without termination. Thus the class of third order diagrams which are of first order in the external field, and which consist of all radiative corections of order $\alpha = e^2/4\pi$ along the electron line, provide a current conserving, gauge invariant set. Vacuum polarization in the external photon line can also be included in this set.

The Feynman diagrams giving the lowest order radiative corrections in electron scattering are then those shown in Fig. 17.2. Here a term in δm_e has been added and subtracted from the starting lagrangian (mass renormalization) so that the free lagangian represents fermions of the correct mass, and an additional interaction lagrangian is then present of the form

$$\delta\mathscr{L} = \delta m_e : \bar{\psi}\psi : \tag{17.6}$$

The contribution of this mass counterterm must then also be included consistently in the Feynman rules.[3] The processes in Fig. 17.2 constitute the radiative corrections through order $\alpha = e^2/4\pi$. We will use the Feynman rules to set up each expression. The Dirac algebra is straightforward. The actual evaluation of the resulting integrals follows from the techniques of Feynman parameterization and four-dimensional momentum integration.[4] These methods are also now discussed in standard texts [Bj65], or courses

[2] The dominant contribution from terms of higher order in the external field consists of Coulomb interactions on the incident and outgoing electron lines. These *Coulomb corrections* imply that one should really use solutions to the Dirac equation in the Coulomb field of the target instead of plane waves for the electron. Though technically complicated, this can be done [Da51, Fe51, Ra54, Gr62, On63, Cu66, Tu68] (an updated version of the appropriate code is available from [He00]). Contributions of second order in the external field where a nuclear target is virtually excited and then de-excited, the so-called *dispersion corrections*, are much more difficult to estimate reliably [Sc55, de66, Fr72b, Do75].

[3] It is assumed that δm_e is normal ordered [Bj65, Fe71] and has a power series expansion $\delta m_{(2)}\, e^2 + \delta m_{(4)}\, e^4 + \cdots$.

[4] Or integration in $n = 4 + \epsilon$ dimensions if one uses dimensional regularization.

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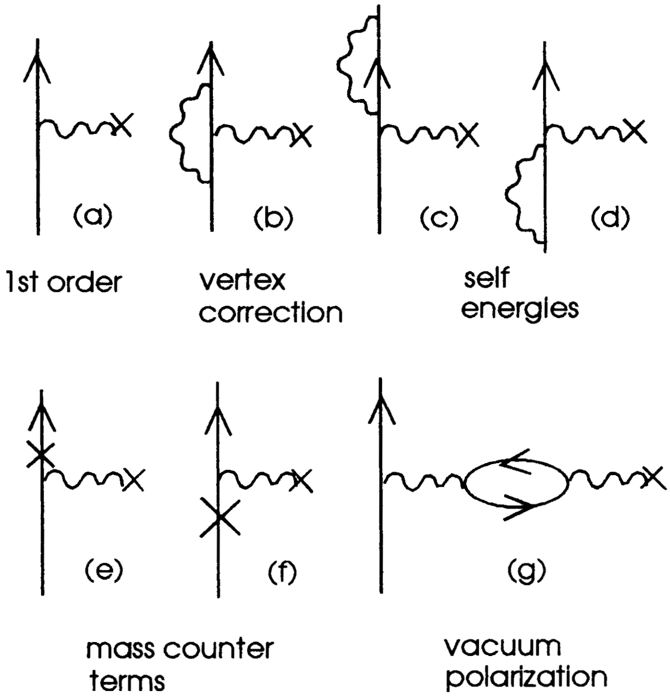

Fig. 17.2. Feynman diagrams for lowest-order radiative corrections in electron scattering.

[Wa91], and it is not the intent to reproduce the derivations. We are primarily concerned here with the results, how they fit together, how they enter into electron scattering, and their interpretation.

Let us consider each component in turn. Consider first the *electron self-energy*. The photon loop and mass counter term corrections to the S-matrix for a free electron are illustrated in Fig. 17.3. The Feynman rules give the S-matrix as[5]

$$S_{fi} = -\frac{(2\pi)^4 i}{\Omega}\delta^{(4)}(k' - k)\bar{u}(k)(\Sigma - \delta m_e)u(k) \tag{17.7}$$

Here the self-energy insertion is defined as

$$\Sigma - \delta m_e = -\frac{ie^2}{(2\pi)^4}\int \frac{d^4q}{q^2}\gamma_\mu \frac{1}{i\gamma_\lambda(k-q)_\lambda + m_e}\gamma_\mu - \delta m_e \tag{17.8}$$

From Lorentz covariance and power counting, this expression can be

[5] For the mass counter term one has the factors $(-1)(-i)\delta m_e$.

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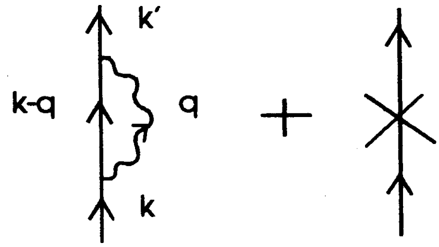

Fig. 17.3. Electron self-energy.

put in the following form

$$\begin{aligned}\Sigma - \delta m_e &= A - \delta m_e \\ &+(ik_\lambda\gamma_\lambda + m_e)B + (ik_\lambda\gamma_\lambda + m_e)\Sigma_f(ik_\sigma\gamma_\sigma + m_e) \quad (17.9)\end{aligned}$$

Here A and B are (infinite) constants independent of k, and Σ_f is finite.

To give mathematical definition to the divergent integral in Eq. (17.8) we introduce a covariant Pauli–Villars regulator which amounts here to replacing the photon propagator by

$$\frac{1}{q^2} \longrightarrow \frac{1}{q^2} - \frac{1}{q^2 + \Lambda^2} \quad (17.10)$$

For very large Λ^2 the second term is negligible, while at fixed Λ^2 the asymptotic behavior of the photon propagator is changed to $\Lambda^2/q^2(q^2 + \Lambda^2)$, and one picks up enough convergence to make the integral finite. Explicit evaluation of the resulting integral on the mass shell, that is for $ik_\lambda\gamma_\lambda + m_e = 0$, yields the mass counter term [Bj65, Wa91]

$$A \equiv \delta m_e = \frac{3\alpha}{2\pi} m_e \left(\ln \frac{\Lambda}{m_e} + \frac{1}{4} \right) \quad (17.11)$$

Consider next *vacuum polarization*. The lowest order vacuum polarization correction to the S-matrix for a free photon as illustrated in Fig. 17.4. The analytic expression is given by

$$S_{fi} = -\frac{(2\pi)^4 i}{\Omega} \delta^{(4)}(l' - l) \frac{1}{\sqrt{4\omega\omega'}} \varepsilon^f_\mu (-\Pi_{\mu\nu}) \varepsilon^i_\nu \quad (17.12)$$

The polarization part is defined by

$$\begin{aligned}\Pi_{\mu\nu} &= -\frac{ie^2}{(2\pi)^4} \int d^4k \,\text{trace} \left[\frac{1}{i\gamma_\lambda(k - l/2)_\lambda + m_e} \gamma_\mu \frac{1}{i\gamma_\sigma(k + l/2)_\sigma + m_e} \gamma_\nu \right] \\ &= (l_\mu l_\nu - l^2 \delta_{\mu\nu}) C(l^2) \quad (17.13)\end{aligned}$$

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Fig. 17.4. Lowest order vacuum polarization correction.

The second relation follows from Lorentz covariance and current conservation. One can write

$$C(l^2) = C(0) - l^2\Pi_f(l^2) \tag{17.14}$$

In order to produce a mathematically well-defined expression, while maintaining current conservation, one can use a more general Pauli–Villars regulator on the *loop integral* in Eq. (17.13)

$$\Pi_{\mu\nu}(l, m_e^2) \to \int g(\lambda^2)d\lambda^2[\Pi_{\mu\nu}(l, m_e^2) - \Pi_{\mu\nu}(l, m_e^2 + \lambda^2)] \tag{17.15}$$

with $g(\lambda^2)$ receiving contributions only from very large $\lambda^2 \approx \Lambda^2$ and

$$\begin{aligned} \int g(\lambda^2)d\lambda^2 &= 1 \\ \int \lambda^2 g(\lambda^2)d\lambda^2 &= 0 \end{aligned} \tag{17.16}$$

One argument in justification of this regularization procedure is that equating higher moments of λ^2 to zero, thereby obtaining additional convergence, will not change the answer. Evaluation of the integrals now results in [Bj65, Wa91]

$$C(0) = \frac{2\alpha}{3\pi} \ln \frac{\Lambda}{m_e} \tag{17.17}$$

Let us denote by e_0 the electric charge used up to this point, i.e. the "bare charge" appearing in the initial lagrangian. If one now combines the lowest order contribution with the vacuum polarization contribution for scattering of an electron in an external field [Fig. 17.2(a), (g)], the result is to change the amplitude in the limit $q^2 \to 0$ from $e_0^2/q^2 \to e^2/q^2$ where the *renormalized charge* is given by

$$e^2 = e_0^2[1 - C(0)] = e_0^2\left(1 - \frac{2\alpha_0}{3\pi} \ln \frac{\Lambda}{m_e}\right) \tag{17.18}$$

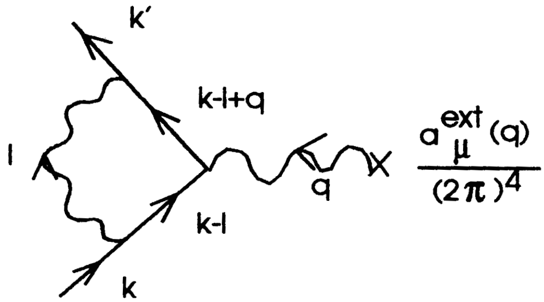

Fig. 17.5. Vertex correction. Here $k \equiv k_1$ and $k' \equiv k_2$.

To this order, *in the radiative corrections*, one can replace $e_0^2 \to e^2$. The vacuum polarization contribution to the above process is now obtained by making the following replacement in the lowest order term

$$\frac{1}{q^2} \longrightarrow \frac{1}{q^2}[1 + q^2\Pi_f(q^2)] \tag{17.19}$$

Here $\Pi_f(q^2)$ is calculated with e^2.

The remaining finite momentum integrals can be evaluated to give the answer in terms of an integral over the Feynman parameter x [Bj65, Wa91]

$$\begin{aligned} l^2\Pi_f(l^2) &= \frac{2\alpha}{\pi}\int_0^1 x(1-x)\ln\left[1 + x(1-x)\frac{l^2}{m_e^2}\right]dx \\ &\to \frac{\alpha}{15\pi}\frac{l^2}{m_e^2} \qquad ; \; l^2 \ll m_e^2 \\ &\to \frac{\alpha}{3\pi}\left[\ln\frac{l^2}{m_e^2} - \frac{5}{3}\right] \qquad ; \; l^2 \gg m_e^2 \end{aligned} \tag{17.20}$$

Consider next the *vertex correction* in Fig. 17.5. The analytic expression for the contribution to the S-matrix is given by

$$S_{fi} = -\frac{e}{\Omega}\bar{u}(k_2)\Lambda_\mu(k_2,k_1)u(k_1)a_\mu^{\text{ext}}(q) \tag{17.21}$$

$$\Lambda_\mu(k_2,k_1) = \frac{ie^2}{(2\pi)^4}\int \frac{d^4l}{l^2}\gamma_\nu \frac{1}{i\gamma_\lambda(k_1 - l + q)_\lambda + m_e}\gamma_\mu \frac{1}{i\gamma_\sigma(k_1 - l)_\sigma + m_e}\gamma_\nu$$

The general form of this vertex follows from Lorentz covariance and power counting as

$$\Lambda_\mu = L\gamma_\mu + \Lambda_{\mu C}(k_2,k_1) \tag{17.22}$$

Here L is a (infinite) constant and the diagonal matrix element of the remaining convergent term, taken between Dirac spinors, vanishes. Regularization of the photon propagator as in Eq. (17.10) again eliminates the

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ultraviolet divergence at high momenta (short wavelengths), and explicit evaluation gives [Bj65, Wa91]

$$L = \frac{\alpha}{2\pi}\left[\ln\frac{\Lambda}{m_e} + \frac{9}{4} - 2\ln\frac{m_e}{\lambda}\right] \tag{17.23}$$

Here, to protect against the *infrared divergence* at low momenta (long wavelengths), the photon has been given a tiny, fictitious mass and the photon propagator has been replaced by

$$\frac{1}{q^2} \longrightarrow \frac{1}{q^2+\lambda^2} \tag{17.24}$$

Note that no physical result can depend on the fictitious photon mass λ^2.

It is relatively easy to evaluate the matrix element of the remaining term in Eq. (17.22) between Dirac spinors $\bar{u}(k_2)\Lambda_{\mu C}(k_2,k_1)u(k_1)$ with the result [Bj65, Wa91]

$$\begin{aligned}
\Lambda_{\mu C} &\doteq F_E(q^2)\gamma_\mu - F_M(q^2)\frac{1}{2m_e}\sigma_{\mu\nu}q_\nu \qquad (17.25)\\
F_M(q^2) &= \frac{\alpha}{\pi}\int_0^1 dx\int_0^x dy\,\frac{m_e^2\,x(1-x)}{m_e^2\,x^2+q^2y(x-y)}\\
F_E(q^2) &= -\frac{\alpha}{2\pi}\int_0^1 dx\int_0^x dy\left\{\ln\left(1+\frac{q^2y(x-y)}{m_e^2\,x^2+\lambda^2(1-x)}\right)\right.\\
&\quad +2m_e^2\left(1-x-\frac{x^2}{2}\right)\\
&\quad \times\left[\frac{1}{m_e^2\,x^2+\lambda^2(1-x)+q^2y(x-y)} - \frac{1}{m_e^2\,x^2+\lambda^2(1-x)}\right]\\
&\quad \left. +q^2(1-x+y)(1-y)\left[\frac{1}{m_e^2\,x^2+\lambda^2(1-x)+q^2y(x-y)}\right]\right\}
\end{aligned}$$

Here $q = k_2 - k_1$, and $\doteq$ means “taken between Dirac spinors.”

The limiting cases of these results are as follows

$$\begin{aligned}
F_M(0) &= \frac{\alpha}{2\pi}\\
F_E(q^2) &= \frac{\alpha}{3\pi}\frac{q^2}{m_e^2}\left(\frac{3}{8}-\ln\frac{m_e}{\lambda}\right) &&;\ q^2 \ll m_e^2\\
&= -\frac{\alpha}{2\pi}\ln\frac{q^2}{m_e^2}\ln\frac{q^2}{\lambda^2} &&;\ q^2 \gg m_e^2 \qquad (17.26)
\end{aligned}$$

Note that the remaining finite part of the vertex $F_E(q^2)$ is infrared divergent; therefore it is *not an observable.*

By direct calculation, one can now establish to $O(\alpha)$ that

$$B = L \tag{17.27}$$

In fact, this relation holds to all orders. It was proven by Ward who observed the general result known as *Ward's Identity* [Wa50]

$$\frac{\partial}{\partial k_\mu}\Sigma^\star(k) = i\Lambda_\mu(k,k) \tag{17.28}$$

Here $\Sigma^\star$ is the proper self-energy and Λ_μ the proper vertex [Bj65, Fe71]. In second order, this result follows immediately from Eqs. (17.8, 17.21). Equation (17.27) can then be derived from it by taking matrix elements between Dirac spinors of identical four-momentum.[6]

When the self-energy insertion $\Sigma - \delta m_e$ is on an external line, the resulting expression obtained from Eqs. (17.8, 17.9) is ambiguous since, for example,

$$-B(i\gamma_\lambda k_\lambda + m_e)\frac{1}{i\gamma_\lambda k_\lambda + m_e}u(k) = -\frac{0}{0}B\,u(k) \tag{17.29}$$

A proper adiabatic limiting procedure says that here the correct answer is to retain $-(B/2)u(k)$, and similarly for the other leg [Bj65, Wa91].

The use of the Fourier transform of Maxwell's equations for the external field allows one to relate that field to its source[7]

$$a_\mu^{\rm ext}(q) = \frac{e_0}{q^2}j_\mu^{\rm ext}(q) \tag{17.30}$$

In *summary*, the addition of all the diagrams in Fig. 17.2 yields to $O(e_0^4)$

$$\begin{aligned} S_{fi} \;=\; & -\frac{e_0^2}{\Omega}\bar{u}(k_2)\left\{\gamma_\mu\left[1 + L - \frac{B}{2} - \frac{B}{2} - C\right] + \gamma_\mu q^2\Pi_f(q^2)\right. \\ & \left. + \Lambda_{\mu C}(k_2,k_1)\right\}u(k_1)\,\frac{1}{q^2}j_\mu^{\rm ext}(q) \end{aligned} \tag{17.31}$$

Ward's identity now leads to an exact cancellation of the term $L - B = 0$. The remaining constant C, *arising entirely from vacuum polarization*, serves to renormalize the charge according to Eq. (17.18). As above, one can then replace $\alpha_0 = \alpha + O(e_0^4)$ to this order in the radiative corrections.

[6] Ward's Identity follows in general by looking at all the Feynman diagrams involved, letting the external electron momentum flow along the electron line, and then differentiating with respect to this momentum.

[7] This relation explicitly exhibits the one additional power of e_0 in the process — i.e., both ends of the vacuum polarization insertion end up on a charge.

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The result is

$$\begin{aligned} S_{fi} &= -\frac{e^2}{\Omega}\bar{u}(k_2)\left\{\gamma_\mu[1+q^2\Pi_f(q^2)] + \Lambda_{\mu C}(k_2,k_1)\right\}u(k_1)\frac{1}{q^2}j_\mu^{\text{ext}}(q) \\ &= -\frac{e}{\Omega}\bar{u}(k_2)\left\{\gamma_\mu[1+F_E(q^2)+q^2\Pi_f(q^2)] - F_M(q^2)\frac{1}{2m_e}\sigma_{\mu\nu}q_\nu\right\} \\ &\quad \times u(k_1)a_\mu^{\text{ext}}(q) \end{aligned} \tag{17.32}$$

Several comments are relevant:

- This amplitude is to be computed with the renormalized charge;
- This result is finite as $\Lambda \to \infty$; there is no longer any ultraviolet divergence;
- The exact second-order (integral) expressions for the quantities appearing in this result are given in Eqs. (17.20, 17.25);
- The presence of form factors in this expression indicates that the electron does indeed have an internal structure; it arises from the interaction with the virtual photon field and is completely calculable within the framework of QED;[8]
- This expression is still infrared divergent in that it depends on the fictitious photon mass λ^2 — hence, as it stands, it is *unobservable*.

[8] There is further internal structure of the electron at much shorter distance scales arising from the weak interactions.