ON CLOSED, TOTALLY BOUNDED SETS

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C. Goffman asserts that "... in a metric space X a set S is compact if and only if it is closed and totally bounded." [1] and "... every totally bounded sequence in a metric space has convergent subsequence." [2].

The statements (incidentally, equivalent to each other) are both wrong, as the following counter-example shows. Take the set of all reals in the open interval $(0, 1)$ with the usual metric. This space is closed and totally bounded, but not compact.

It is well-known that the statements are true under the additional hypothesis of completeness of the space. In this paper, we prove that the statements are true <u>only when</u> the space is complete.

THEOREM. For a metric space (X, d) the following statements are equivalent:

(1) (X, d) is complete.

(2) Every closed and totally bounded subspace of X is compact.

<u>Proof.</u> (1) implies (2) is known. (see e.g. [3]).

(2) $\Longrightarrow$ (1): Let s be a cauchy sequence in X, and suppose s has no cluster points. Let $A = \{s(n) \mid n = 1, 2, \ldots\}$ be the range of the sequence. Then (i) A is infinite, otherwise s would have a cluster point; (ii) A is closed, since A has no cluster points; (iii) A is totally bounded, because for any $\epsilon > 0$, there exists an N such that the ϵ-neighborhood of $s(N)$ contains all but finitely many points of A. By hypothesis, A is compact and hence has the Bolzano-Weierstrass property. Since A is infinite, A must have a cluster point, a contradiction. Thus every cauchy sequence in X has a cluster point and the space is complete.

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