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Lower hybrid current drive for an intense monochromatic applied field

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The intense applied lower hybrid electric fields used to drive current in tokamaks can result in the formation of velocity space island structure. When this happens the lower hybrid current drive efficiency can be calculated for a monochromatic wave and is shown to be below the quasilinear level.

Keywords: plasma heating, plasma nonlinear phenomena, plasma waves

1. Introduction

Quasilinear (QL) descriptions of lower hybrid current drive (LHCD) have successfully predicted and explained how non-inductive currents can be driven in tokamaks (Fisch 1978; Karney & Fisch 1979, 1985; Fisch & Boozer 1980; Fisch & Karney 1981; Cordey *et al*. 1982; Antonsen & Chu 1982; Taguchi 1983), although discrepancies remain (Bonoli 2014). The intense applied radio frequency (RF) fields used to drive the current result in departures of the electron distribution function from Maxwellian, indicating a failure of QL theory (Catto & Tolman 2021*a*,*b*). The purpose here is to investigate the failure of QL theory and present an alternative description valid for a very intense applied monochromatic lower hybrid wave. A brief analysis of lower hybrid waves appears in the appendix of Catto & Zhou (2023).

Before presenting the detailed calculation, it may be useful to provide a few more background details. The pioneering evaluations of LHCD in the late 1970s and early 1980s were for plane waves in homogeneous magnetised plasmas. The QL descriptions used seldom mentioned the need to retain collisions in the linearised kinetic equation, but when the results were applied to tokamak geometry it was normally assumed that successive passes through resonance were uncorrelated. Catto (2020) first realised that the resonant particles were particularly sensitive to collisions and the narrow collisional boundary layer formed that enhanced the role of collisions for them. Subsequent work by Catto & Tolman (2021*a*,*b*) extended the collisional treatment to tokamak geometry, and demonstrated that the resonance condition is not local but rather a transit averaged resonance condition. The charges satisfying or nearly satisfying the transit averaged resonance condition are those that are particularly sensitive to collisions which act to decorrelate successive poloidal transits. To lowest order the collision frequency does not enter in QL treatments, but its absence does not mean collisions are not playing a role. The structure of the Fokker–Planck equation for steady state applied wave fields requires collisions to resolve the singular behaviour in the linearised kinetic equation since initial conditions must be unimportant. To demonstrate the role of tokamak geometry Catto & Zhou (2023) considered LHCD and used the adjoint procedure of Antonsen & Chiu (1982) with the Cordey *et al*. (1982) eigenfunctions to get a rigorous result for the trapped fraction corrections to LHCD and to demonstrate that in their absence the usual result was recovered.

To understand why QL theory fails, only a simple nonlinear electron drift kinetic equation (in $v, v_{||}$ variables) for a monochromatic applied parallel electric field $E_{||} = \tilde{E}_{||}\sin(\omega t - k_{||}z)$ of frequency ω and parallel wavenumber $k_{||}$ needs to be considered, namely

$$\partial f_1/\partial t + v_{||}\partial f_1/\partial z - (e/m)E_{||}\vec{z}\cdot\nabla_v(f_0 + f_1) = C\{f_1\}. \tag{1.1}$$

Here $f = f_0 + f_1$ with $f_0(v)$ a Maxwellian and $f_1(z, v, v_{||}, t)$ a small correction, $\vec{B} = B\vec{z}$ is the $\vec{z}$ directed constant magnetic field, $C\{f_1\}$ is the linearised electron collision operator and e and m are the proton charge and electron mass. Defining a resonance width $\Delta v_{||}$ via

$$k_{||}v_{||} - \omega = k_{||}\Delta v_{||} \tag{1.2}$$

gives an estimate of f_1 by balancing the drive and resonance terms:

$$f_1 k_{||}\Delta v_{||} \sim \left(e\tilde{E}_{||}v_{||}/mv\right)\partial f_0/\partial v. \tag{1.3}$$

Resonant electrons are particularly sensitive to collisions. Estimating

$$C\{f_1\} \sim \nu_e v_e^2 \partial^2 f_1/\partial v_{||}^2 \sim \nu_e v_e^2 f_1/\left(\Delta v_{||}\right)_\nu^2, \tag{1.4}$$

with ν_e and v_e the electron collision frequency and thermal speed, and balancing collisions with the resonant term leads to the normalised collisional boundary layer width

$$\left(\Delta v_{||}\right)_\nu/v_e \sim |\nu_e/k_{||}v_e|^{1/3} \ll 1 \tag{1.5}$$

and effective resonant electron collision frequency $\nu_{\mathrm{eff}} = \nu_e v_e^2/(\Delta v_{||})_\nu^2 \sim \nu_e(k_{||}v_e/\nu_e)^{2/3} \gg \nu_e$.

The nonlinear term ignored in QL theory cannot be neglected when

$$(v_{||}/v)\partial f_0/\partial v \sim \partial f_1/\partial v_{||} \sim f_1/\Delta v_{||}. \tag{1.6}$$

Inserting (1.3) into (1.6) gives a normalised island width estimate of

$$\left(\Delta v_{||}\right)_{\mathrm{is}}/v_e \sim \left|e\tilde{E}_{||}/mk_{||}v_e^2\right|^{1/2}. \tag{1.7}$$

QL theory assumes $(\Delta v_{||})_\nu \gg (\Delta v_{||})_{\mathrm{is}}$ or $|\nu_e/k_{||}v_e|^{1/3} \gg |e\tilde{E}_{||}/mk_{||}v_e^2|^{1/2}$. However, for fields intense enough to cause the electron distribution function to develop significant non-Maxwellian features this inequality is violated. To verify this, the usual QL diffusivity can be estimated by using $D_{\mathrm{ql}} \sim (e\tilde{E}_{||}/m)^2\delta(\omega - k_{||}v_{||}) \sim (e\tilde{E}_{||}/m)^2/k_{||}(\Delta v_{||})_\nu$, corresponding to the delta function estimate $\delta(\omega - k_{||}v_{||}) \sim 1/k_{||}(\Delta v_{||})_\nu$. QL diffusion modifies the electron Maxwellian once the QL and collision operators become comparable, that is, once $D_{\mathrm{ql}}f_0 \sim \nu_e v_e^2 f_0$ or when

$$\left(\Delta v_{||}\right)_\nu \sim \left(\Delta v_{||}\right)_{\mathrm{is}}. \tag{1.8}$$

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This condition is the same as found by allowing the nonlinear, drive and resonant terms to be of the same order, $f_1 k_{||}(\Delta v_{||}) \sim (e\tilde{E}_{||} v_{||}/mv)\partial f_0/\partial v \sim (e\tilde{E}_{||}/m) f_1/\Delta v_{||}$, suggesting QL theory is no longer valid. Moreover, it indicates that finite velocity space island effects should be retained because the RF fields are too intense for a QL treatment.

The ordering $(\Delta v_{||})_\nu \sim (\Delta v_{||})_{\text{is}}$ is not analytically tractable (Hamilton *et al.* 2023; Catto 2024). Consequently, the focus of the following sections is the $(\Delta v_{||})^3_{\text{is}} \gg (\Delta v_{||})^3_\nu$ limit. The next section introduces notation and derives the nonlinear equation to be solved for a very intense monochromatic wave. Section 3 solves this nonlinear kinetic equation when $(\Delta v_{||})^3_{\text{is}} \gg (\Delta v_{||})^3_\nu$. In § 4 the lower hybrid driven parallel current and power absorbed to drive the current are evaluated, and the resulting current drive efficiency found. The result is compared with the standard QL expression. The final section is a brief discussion.

2. Nonlinear electron kinetic equation

Consider an applied lower hybrid wave of frequency $\omega \ll \Omega = eB/mc$ in a uniform plasma with a constant magnetic field $\vec{B} = B\vec{z}$, where B is the magnitude of the magnetic field and $\vec{z}$ is the unit vector in the direction of the uniform field. For a lowest order Maxwellian distribution function,

$$f_0 = n(m/2\pi T)^{3/2} e^{-mv^2/2T}, \tag{2.1}$$

the perturbed distribution function f_1 satisfies the nonlinear electron drift kinetic equation

$$\partial f_1/\partial t + v_{||}\partial f_1/\partial z - (e/m)E_{||}\partial f_1/\partial v_{||} - C\{f_1\} = -(e/T)E_{||}v_{||}f_0 \tag{2.2}$$

in $v_{||}$ and $v_\perp$ velocity variables with $v^2 = v_{||}^2 + v_\perp^2$. Here n and T are the electron density and temperature, and $f_0 + f_1 = f$ with $f_0 \gg f_1$. The perturbing parallel applied electric field of the lower hybrid wave is taken to be $\vec{E} = E_{||}\vec{z}$, with the unimportant unperturbed electric field neglected for simplicity.

The collision operator is needed to resolve singular behaviour at a resonance or in the vicinity of small scale velocity space structure. It is approximated by the usual like and unlike terms, namely the high speed expansion ($v^2 \gg v_e^2 = 2T/m$), self adjoint operator

$$\begin{aligned} C\{f\} &= \nabla_v \cdot \left\{ \frac{\nu_e}{2x^3}\left[\left(v^2 \overleftrightarrow{I} - \vec{v}\vec{v}\right)\cdot\nabla_v f_1 + \frac{2Tf_0}{(Z+1)m}\nabla_v\left(\frac{f_1}{f_0}\right)\right]\right\} \\ &\approx \frac{\nu_e}{2x^3}\left[v_\perp^2 + \frac{v_e^2}{(Z+1)}\right]\frac{\partial^2 f_1}{\partial v_{||}^2}, \end{aligned} \tag{2.3}$$

where $x = v/v_e$, $\nu_e = 3\sqrt{\pi}(Z+1)\nu_{ee}/4$ and $\nu_{ee} = \nu_{ei}/Z = 4\sqrt{2\pi}e^4 n \ell\text{n}\,\Lambda_c/3m^{1/2}T^{3/2}$ for a quasineutral plasma, with Z the ion charge number and $\ell\text{n}\,\Lambda_c$ the Coulomb logarithm. The final form of $C\{f\}$ is all that is required near the resonance. The high speed expansion of the collision operator was used for the original LHCD calculations (Karney & Fisch 1979, 1985) and was also used for a recent linearised treatment of LHCD in a tokamak (Catto & Zhou 2023) which found that $\omega^2/k_{||}^2 v_e^2 \approx 5/2$ maximises current drive.

To make analytic progress the applied electric field is assumed to be monochromatic with a wave frequency $\omega > 0$, and a parallel wave vector $k_{||} > 0$:

$$E_{||} = \tilde{E}_{||}\sin(k_{||}z - \omega t). \tag{2.4}$$

If only the linearised equation for f_1 were to be solved as in standard QL theory, it would be adequate to let $f_1 = \mathfrak{Im}[\tilde{f} \mathrm{e}^{i(k_{||}z-\omega t)}]$, with $\mathfrak{Im}$ denoting imaginary part, and solve

$$\mathrm{i}\big(\omega - k_{||}v_{||}\big)\tilde{f} + \nu v_{\perp z}^2 \partial^2 \tilde{f}/\partial v_{||}^2 = (e/T)\tilde{E}_{||} v_{||} f_0 \approx \big(e\omega\tilde{E}_{||}/Tk_{||}\big) f_0, \tag{2.5}$$

where only pitch angle scattering need be retained to resolve the singularity at $\omega = k_{||}v_{||}$, $v_{\perp z}^2 = v_\perp^2 + v_e^2/(Z+1)$ and $\nu = \nu_e/2x^3$. The solution is (Su & Oberman 1968; Johnston 1971; Auerbach 1977; Catto 2020; Catto & Tolman 2021*b*)

$$\tilde{f} = -\frac{e\omega\tilde{E}_{||} f_0}{T\left(k_{||}^5 v_{\perp z}^2 \nu\right)^{1/3}} \int_0^\infty \mathrm{d}\tau\, \mathrm{e}^{-\mathrm{i}s\tau - \tau^3/3}, \tag{2.6}$$

with $s = (k_{||}/v_{\perp z}^2\nu)^{1/3}(v_{||} - \omega/k_{||})$ and $s = 1$ giving a velocity space collisional boundary layer width of $v_{||} - \omega/k_{||} = (\Delta v_{||})_\nu = \left|\nu v_{\perp z}^2/k_{||}\right|^{1/3}$.

When the applied lower hybrid wave amplitude becomes large enough, the non-linear term cannot be neglected and the full nonlinear drift kinetic equation must be considered by letting $f = f_0 + f_1$. Fortunately, f_0 is slowly varying as is the $v_\perp$ dependence of f_1 because the resonance involves only $v_{||}$. Therefore, the ordering

$$\partial f_1/\partial v_{||} \sim \partial f_0/\partial v_{||} \tag{2.7}$$

is employed. Assuming a homogeneous plasma such that

$$f_1 = f_1(\phi, v_{||}, v_\perp), \tag{2.8}$$

with $v_\perp$ entering as a parameter, and defining

$$\phi = k_{||}z - \omega t \tag{2.9}$$

leads to

$$(k_{||}v_{||} - \omega)\partial f_1/\partial\phi - (e\tilde{E}_{||}/m)\sin\phi\,\partial(f_0 + f_1)/\partial v_{||} = C\{f_1\}. \tag{2.10}$$

Now only the last form of (2.3) matters. Letting $v_{||} = \omega/k_{||} + u$, with u small, leads to

$$k_{||}u\frac{\partial f_1}{\partial\phi} - \frac{e\tilde{E}_{||}}{m}\sin\phi\left(\frac{\partial f_0}{\partial v_{||}} + \frac{\partial f_1}{\partial u}\right) = \nu v_{\perp z}^2\frac{\partial^2 f_1}{\partial u^2}. \tag{2.11}$$

To cast this equation into the form considered by Hamilton *et al.* (2023), let

$$f_1 = g(u, \phi) - (u - \sigma\alpha)\partial f_0/\partial v_{||}, \tag{2.12}$$

where $\sigma = u/|u| = \pm 1$ or 0, and α is a constant to be determined, to find

$$k_{||}u\frac{\partial g}{\partial\phi} - \frac{e\tilde{E}_{||}}{m}\sin\phi\frac{\partial g}{\partial u} = \nu v_{\perp z}^2\frac{\partial^2 g}{\partial u^2}. \tag{2.13}$$

Letting

$$j = \left|m/e\tilde{E}_{||}k_{||}\right|^{1/2}k_{||}u = \left|m/e\tilde{E}_{||}k_{||}\right|^{1/2}\big(k_{||}v_{||} - \omega\big) \tag{2.14}$$

and

$$\varDelta = \nu k_{||}^2 v_{\perp z}^2\left|\frac{m}{e\tilde{E}_{||}k_{||}}\right|^{3/2} > 0 \tag{2.15}$$

gives the equation to be the same as that solved numerically by Hamilton *et al.* (2023):

$$j\frac{\partial g}{\partial \phi} - \sin\phi\frac{\partial g}{\partial j} = \Delta\frac{\partial^2 g}{\partial j^2}. \tag{2.16}$$

For $j \sim 1$, (2.14) gives the velocity space island width to be

$$\left(\Delta v_{||}\right)_{\text{is}} = \left|\frac{e\tilde{E}_{||}}{mk_{||}}\right|^{1/2} \ll \frac{\omega}{k_{||}}. \tag{2.17}$$

QL theory assumes that the collisional boundary layer width

$$\left(\Delta v_{||}\right)_{\nu} = \left|\nu v_{\perp z}^2/k_{||}\right|^{1/3} \tag{2.18}$$

is wider than this velocity space island structure, requiring $\varDelta = [(\Delta v_{||})_\nu/(\Delta v_{||})_{\text{is}}]^3 \gg 1$. However, for a strong applied RF field at low enough density the QL assumption (2.17) fails. In the next section an analytic solution to (2.16) is found in the intense-field limit $\varDelta \ll 1$.

3. Solution of the kinetic equation in the intense field limit

The limit where the velocity space island and collisional boundary layer widths are comparable ($\varDelta \sim 1$) is not analytically tractable. However, when the velocity space island width is small, $|(\Delta v_{||})_{\text{is}}/v_e|^3 \ll 1$, but larger than the collisional boundary layer width,

$$1 \gg \left|\frac{e\tilde{E}_{||}}{mk_{||}v_e^2}\right|^{3/2} \gg \frac{\nu}{k_{||}v_e}, \tag{3.1}$$

it becomes possible to analytically investigate the failure of QL treatments of LHCD at low density, high temperature and large applied wave amplitudes.

The preceding nonlinear kinetic equation (2.16) with a temporal evolution term has been solved numerically by Hamilton *et al.* (2023) for an astrophysical application. They also provided a partial analytic steady solution for $\varDelta \ll 1$, which was completed by Catto (2024) for a stellarator transport evaluation. The solution is skew symmetric, satisfying

$$g(j,\phi) = -g(-j,-\phi). \tag{3.2}$$

The analytic treatment starts by introducing the reduced constant of the motion or Hamiltonian h defined by

$$h(j,\phi) = j^2/2 - \cos\phi \tag{3.3}$$

with $h = 1$ the separatrix between the bound or librating ($-1 < h < 1$) and the unbound or circulating ($h > 1$) electrons (to avoid confusion, the terminology trapped and passing is avoided and reserved for magnetic wells). The reduced Hamiltonian allows the kinetic equation to be rewritten in terms of h, ϕ variables as

$$\left.\frac{\partial g}{\partial \phi}\right|_h = \Delta\left.\frac{\partial}{\partial h}\right|_\phi\left(j\left.\frac{\partial g}{\partial h}\right|_\phi\right). \tag{3.4}$$

Then $\varDelta \ll 1$ suggests a solution of the form $g = g_1(h) + g_2(h,\phi) + \cdots$ to satisfy $\partial g_1/\partial\phi|_h = 0$ to lowest order. The next order equation,

$$\left.\frac{\partial g_2}{\partial \phi}\right|_h = \Delta\left.\frac{\partial}{\partial h}\right|_\phi\left(j\left.\frac{\partial g_1}{\partial h}\right|_\phi\right), \tag{3.5}$$

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then leads to the solubility constraint from the collision operator, namely

$$\left.\frac{\partial}{\partial h}\right|_{\phi}\left[\left(\oint_{h}\mathrm{d}\phi j\right)\left.\frac{\partial g_1}{\partial h}\right|_{\phi}\right]=0. \tag{3.6}$$

Therefore, g_1 is independent of collision frequency, but its form is constrained by collisions.

For the bound or librating electrons $g_1=0=\alpha=\sigma$, giving $f_1=u\partial f_0/\partial v_{||}$ as the only acceptable and well behaved solution. For the freely circulating or unbound electrons, far from the resonance, a solution with $g_1\rightarrow(u-\sigma\alpha)\partial f_0/\partial v_{||}$ is required, where $\sigma=\pm u/|u|=\pm j/|j|$ and α is a constant still to be determined. Introducing the complete elliptic function of the second kind, E, leads to (Hamilton *et al.* 2023; Catto 2024)

$$\begin{aligned} g_1&=\sigma\pi\left|\frac{eE_{||}}{mk_{||}}\right|^{1/2}\frac{\partial f_0}{\partial v_{||}}\int_k^1\frac{\mathrm{d}t}{t^2E\,(t)}\underset{k\rightarrow 0}{\rightarrow}\sigma\left|\frac{eE_{||}}{mk_{||}}\right|^{1/2}\frac{\partial f_0}{\partial v_{||}}\left[\frac{2}{k}-1.379-\frac{k}{2}+O(k^3)\right]\\ &\approx\left|\frac{eE_{||}}{mk_{||}}\right|^{1/2}\frac{\partial f_0}{\partial v_{||}}\left[j-1.379\sigma-\frac{\cos\phi}{j}\right]=\left(u-\sigma\alpha-\frac{e\tilde{E}_{||}\cos\phi}{mk_{||}u}\right)\frac{\partial f_0}{\partial v_{||}}, \end{aligned} \tag{3.7}$$

with $j=\sqrt{2(h+\cos\phi)}=(2/k)\sqrt{1-k^2\sin^2(\phi/2)}$, $k=\sqrt{2/(h+1)}$, $\int_{-\pi}^{\pi}\mathrm{d}\varphi j=\sigma 8k^{-1}E(k)$ and $\alpha=1.379$ now determined. The unbound solution satisfies $g_1\rightarrow 0$ at the separatrix ($h=1$), but $\partial g_1/\partial h|_{\phi}$ and f_1 step across it. The narrow collisional boundary layer about the separatrix provides the smooth matching as in the banana regime of neoclassical theory. Moreover, far from the resonance layer $f_1(k\rightarrow 0)\rightarrow 0$.

Summarising, the full solution to lowest order (Hamilton *et al.* 2023; Catto 2024) is

$$f_1=\left|\frac{eE_{||}}{mk_{||}}\right|^{1/2}\frac{\partial f_0}{\partial v_{||}}\left[\sigma\pi\int_k^1\frac{\mathrm{d}\tau}{\tau^2E\,(\tau)}-(j-1.379\sigma)\right], \tag{3.8}$$

but now with $\sigma=\pm u/|u|=\pm 1$ for the unbound or circulating electrons and $\sigma=0$ for the bound or librating electrons. Even though the solution is independent of the collision frequency, its form is determined by the solubility constraint from the collision operator. The 1.379σ step in the solution f_1 at the separatrix is not cancelled by the piecewise continuous behaviour of the derivative $\partial g_1/\partial h|_{\phi}$ there. Instead, the step is smoothed by a narrow collisional boundary layer at the separatrix that need not be resolved by the procedure here. The behaviour in the boundary layer does not play a role in the results that follow next.

4. Driven current and power absorbed in the intense field limit

The skew symmetric perturbed solution $f_1(u,\phi)=-f_1(-u,-\phi)$ cannot lead to a spatiotemporal averaged density as seen by letting $u\rightarrow -u$ and $\phi\rightarrow-\phi$, then applying $\langle\cdots\rangle_{\phi}=\oint\mathrm{d}\phi(\cdots)/2\pi$ at fixed $v_{||}$ and employing $d^3v=2\pi v_{\perp}\mathrm{d}v_{\perp}\mathrm{d}v_{||}\propto\mathrm{d}u\propto\mathrm{d}j$, to form

$$\left\langle\int_{-\infty}^{\infty}\mathrm{d}uf_1(u,\phi)\right\rangle_{\phi}=-\left\langle\int_{-\infty}^{\infty}\mathrm{d}uf_1(-u,-\phi)\right\rangle_{\phi}=-\left\langle\int_{-\infty}^{\infty}\mathrm{d}uf_1(u,\phi)\right\rangle_{\phi}=0, \tag{4.1}$$

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where the limits of the u integration are for the outer limit of the inner-boundary-layer region and, of course, not actually infinite. However, the parallel current requires the u moment

$$\left\langle \int_{-\infty}^{\infty} \mathrm{d}u u f_1(u,\phi) \right\rangle_\phi = -\left\langle \int_{-\infty}^{\infty} \mathrm{d}u u f_1(-u,-\phi) \right\rangle_\phi = 2\left\langle \int_{0}^{\infty} \mathrm{d}u u f_1(u,\phi) \right\rangle_\phi \neq 0 \tag{4.2}$$

and the extra u means that skew symmetry allows a parallel current! As a result, the driven parallel current is evaluated using

$$\langle J_{||}\rangle_\phi = -e\left\langle \int d^3 v v_{||} f_1 \right\rangle_\phi = -e\left\langle \int d^3 v u f_1 \right\rangle_\phi . \tag{4.3}$$

Only the circulating electrons carry current. Moreover, integrating by parts (recalling $j^2 f_1 \propto 1/j \to 0$ far from the separatrix as shown in (3.7)) results in

$$\begin{aligned} \langle J_{||}\rangle_\phi &= -\frac{e}{2}\left\langle \int d^3 v f_1 \frac{\partial u^2}{\partial u} \right\rangle_\phi = \frac{e}{2}\left\langle \int d^3 v u j \left.\frac{\partial f_1}{\partial j}\right|_\phi \right\rangle_\phi \\ &= \frac{2\pi e^2}{m}\left|\frac{\tilde{E}_{||}}{k_{||}}\right| \left\langle \int_0^\infty \mathrm{d}v_\perp v_\perp \int_0^\infty \mathrm{d}j j^2 \left.\frac{\partial f_1}{\partial j}\right|_\phi \right\rangle_\phi , \end{aligned} \tag{4.4}$$

where both signs of sigma from j are summed over. Inserting

$$\left.\frac{\partial f_1}{\partial j}\right|_\phi = -\left|\frac{eE_{||}}{mk_{||}}\right|^{1/2} \frac{\partial f_0}{\partial v_{||}} \left[\frac{\sigma\pi}{k^2 E\,(k)} \left.\frac{\partial k}{\partial j}\right|_\phi + 1\right], \tag{4.5}$$

leads to

$$\langle J_{||}\rangle_\phi = \frac{2en\omega \mathrm{e}^{-\omega^2/k_{||}^2 v_e^2}}{\pi^{1/2}|k_{||}|} \left|\frac{eE_{||}}{mk_{||}v_e^2}\right|^{3/2} \left\langle \int_0^\infty \mathrm{d}j j^2 \left[\left.\frac{\partial k}{\partial j}\right|_\phi \frac{\pi}{k^2 E\,(k)} + 1\right]\right\rangle_\phi . \tag{4.6}$$

Because the j derivative of k is at fixed ϕ, the ϕ integral can be performed at fixed k or h by using $\langle j^2\rangle_\phi = 2h$ and noting $\partial k/\partial j|_\phi < 0$ to find

$$\left\langle \int_0^\infty \mathrm{d}j j^2 \left.\frac{\partial k}{\partial j}\right|_\phi \frac{\pi}{k^2 E\,(k)} \right\rangle_\phi = -\pi \int_0^1 \mathrm{d}k \frac{\langle j^2\rangle_\phi}{k^2 E\,(k)} = -2\pi \int_0^1 \mathrm{d}k \frac{(2-k^2)}{k^4 E\,(k)}. \tag{4.7}$$

This procedure removes the awkwardness of the upper limit of the j integral in the outer limit of the inner region. A similar procedure and use of $\langle j\rangle_\phi = 4\pi^{-1}k^{-1}E(k)$ gives

$$\left\langle \int_0^\infty \mathrm{d}j j^2 \right\rangle_\phi = \int_1^\infty \mathrm{d}h\, \langle j\rangle_\phi = 4\int_0^1 \mathrm{d}k k^{-3}\, \langle j\rangle_\phi = \frac{16}{\pi}\int_0^1 \mathrm{d}k k^{-4} E(k). \tag{4.8}$$

As a result, the parallel current is negative (since $\omega/k_{||} > 0$) and given by

$$\langle J_{||}\rangle_\phi = -\frac{16en\omega}{\sqrt{\pi}|k_{||}|}\left|\frac{e\tilde{E}_{||}}{mk_{||}v_e^2}\right|^{3/2} \mathrm{e}^{-\omega^2/k_{||}^2 v_e^2} \int_0^1 \frac{\mathrm{d}k}{k^4}\left[\frac{\pi(2-k^2)}{4E\,(k)} - \frac{2E\,(k)}{\pi}\right], \tag{4.9}$$

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where

$$C=\int_0^1 \frac{\mathrm{d}k}{k^4}\left[\frac{\pi(2-k^2)}{4E\,(k)}-\frac{2E\,(k)}{\pi}\right]=0.047675. \tag{4.10}$$

Therefore, the lower hybrid driven current for an intense applied field is

$$\left\langle J_{||}\right\rangle_{\phi}=-0.430env_e\left|\frac{\omega}{k_{||}v_e}\right|\left|\frac{e\tilde{E}_{||}}{mk_{||}v_e^2}\right|^{3/2}\mathrm{e}^{-\omega^2/k_{||}^2v_e^2}. \tag{4.11}$$

Unlike the QL limit for which $\langle J_{||}\rangle_\phi \propto 1/\nu_e$, the expression for $\langle J_{||}\rangle_\phi$ is independent of the collision frequency and scales as $\tilde{E}_{||}^{3/2}$.

The RF power absorbed by the electrons is proportional to ν_e, as can be seen by first using $2\sin\phi=-\mathrm{d}j^2/d\phi|_h$ at fixed h to write

$$\begin{aligned}P=\left\langle E_{||}J_{||}\right\rangle_\phi &= -e\tilde{E}_{||}\left\langle\sin\phi\int d^3v v_{||}f\right\rangle_\phi \approx \frac{\omega e\tilde{E}_{||}}{2k_{||}}\left\langle\int d^3vf\left.\frac{\mathrm{d}j^2}{\mathrm{d}\phi}\right|_h\right\rangle_\phi \\ &= -\frac{\omega e\tilde{E}_{||}}{2k_{||}}\left\langle\int d^3vj^2\left.\frac{\partial g_2}{\partial\phi}\right|_h\right\rangle_\phi = -\frac{\omega e\tilde{E}_{||}}{2k_{||}}\left\langle\int d^3v\Delta j^2\left.\frac{\partial}{\partial h}\right|_\phi\left(j\left.\frac{\partial g_1}{\partial h}\right|_\phi\right)\right\rangle_\phi.\end{aligned} \tag{4.12}$$

In the preceding, $v_{||}=\omega/k_{||}+u$ and skew symmetry give $\langle\int_{-\infty}^{\infty}\mathrm{d}uu\sin\phi f_1(u,\phi)\rangle_\phi=0$ and

$$\begin{aligned}\left\langle\int_{-\infty}^{\infty}\mathrm{d}u\sin\phi f_1(u,\phi)\right\rangle_\phi &= -\left\langle\int_{-\infty}^{\infty}\mathrm{d}u\sin\phi f_1(-u,-\phi)\right\rangle_\phi \\ &= 2\left\langle\int_0^{\infty}\mathrm{d}u\sin\phi f_1(u,\phi)\right\rangle_\phi \neq 0.\end{aligned} \tag{4.13}$$

Then $u\propto j$, $j\mathrm{d}j=\mathrm{d}h$ at fixed ϕ and $h \underset{h\to\infty}{\longrightarrow} 2/k^2$ yield

$$\begin{aligned}\left\langle\int_{-\infty}^{\infty}\mathrm{d}jj^2\left.\frac{\partial}{\partial h}\right|_\phi j\left.\frac{\partial g_1}{\partial h}\right|_\phi\right\rangle_\phi &= 2\left\langle\int_1^{\infty}\mathrm{d}hj\left.\frac{\partial}{\partial h}\right|_\phi j\left.\frac{\partial g_1}{\partial h}\right|_\phi\right\rangle_\phi \\ &= 2\int_1^{\infty}\mathrm{d}h\left[\left.\frac{\partial}{\partial h}\right|_\phi\langle j^2\rangle_\phi\left.\frac{\partial g_1}{\partial h}\right|_\phi-\left.\frac{\partial g_1}{\partial h}\right|_\phi\right] \\ &= 4h\frac{\partial g_1}{\partial h}\Big|_{\phi,h=1}^{h\to\infty}-2g_1\Big|_{h=1}^{h\to\infty} \\ &= 2\left|\frac{e\tilde{E}_{||}}{mk_{||}}\right|^{1/2}\frac{\partial f_0}{\partial v_{||}}\left[\frac{\pi}{2}-\frac{2}{k}+\left(\frac{2}{k}-1.379\right)\right] \\ &= 0.384\left|\frac{e\tilde{E}_{||}}{mk_{||}}\right|^{1/2}\left.\frac{\partial f_0}{\partial v_{||}}\right|_{u=0}\end{aligned} \tag{4.14}$$

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Consequently, using $\mathrm{d}v_{||} = \mathrm{d}u = |e\tilde{E}_{||}/mk_{||}|^{1/2}\mathrm{d}j$ yields

$$P = -0.384\pi m\omega \left|\frac{e\tilde{E}_{||}}{mk_{||}}\right|^2 \int_0^\infty \mathrm{d}v_\perp v_\perp \Delta \left.\frac{\partial f_0}{\partial v_{||}}\right|_{u=0}$$
$$= 0.384\frac{mn\omega^2 \nu_e \mathrm{e}^{-\omega^2/k_{||}^2 v_e^2}}{\pi^{1/2}k_{||}^2 v_e^2}\left|\frac{e\tilde{E}_{||}}{m}\right|^{1/2}\int_0^\infty \mathrm{d}v_\perp \frac{v_\perp v_{\perp z}^2 \mathrm{e}^{-v_\perp^2/v_e^2}}{\left(v_\perp^2 + \omega^2/k_{||}^2\right)^{3/2}}. \tag{4.15}$$

Recalling that the collision operator is valid for $x^2 = v^2/v_e^2 \gg 1$, while $\omega^2/k_{||}^2 v_e^2 \approx 5/2 \gg 1$ and $v_\perp^2 \sim v_e^2$, the resulting integral is approximately

$$\int_0^\infty \mathrm{d}v_\perp \frac{v_\perp v_{\perp z}^2 e^{-v_\perp^2/v_e^2}}{\left(v_\perp^2 + \omega^2/k_{||}^2\right)^{3/2}} \approx \frac{|k_{||}|^3 v_e^4}{2\omega^3}\left(\frac{Z+2}{Z+1}\right). \tag{4.16}$$

As a result, the RF power absorbed to drive the current for an intense applied lower hybrid wave is

$$P = 0.108\frac{Z+2}{Z+1}mnv_e^2\nu_e \left|\frac{k_{||}v_e}{\omega}\right| \left|\frac{e\tilde{E}_{||}}{k_{||}mv_e^2}\right|^{1/2} \mathrm{e}^{-\omega^2/k_{||}^2 v_e^2}. \tag{4.17}$$

The driven current of (4.11) only depends on g_1 as seen by (4.4). It only depends on the reduced constant of the motion h from (3.4). The form for g_1 is constrained by the collision operator to satisfy (3.6), but the collision frequency does not enter. However, to drive current there must be dissipation. It enters through the power absorbed by the electrons from the applied lower hybrid wave. Collisions enter g_2, as seen by (3.5), and it is required in (4.12).

Using $\nu_e = 3\pi^{1/2}(Z+1)\nu_{ee}/4$, the current drive efficiency in the intense field limit is

$$\frac{\left|\langle J_{||}\rangle_\phi \tilde{E}_{||}\right|}{P} = \frac{2.99\omega}{(Z+2)\nu_{ee}}\left|\frac{\omega}{k_{||}v_e}\right|\left|\frac{e\tilde{E}_{||}}{mk_{||}v_e^2}\right|^2 \tag{4.18}$$

leading to the normalised form

$$\frac{\left|\langle J_{||}\rangle_\phi\right|/env_e}{P/mnv_e^2\nu_{ee}} = \frac{2.99\omega^2}{(Z+2)k_{||}^2 v_e^2}\left|\frac{e\tilde{E}_{||}}{mk_{||}v_e^2}\right|. \tag{4.19}$$

Ignoring aspect ratio modifications (Catto & Muni 2023) , the usual QL result (Fisch 1978) is

$$\frac{J_{||}^{LH}/env_e}{P_{cd}^{LH}/mnv_e^2\nu_{ee}} = \frac{3.01\omega^2}{(Z+5)k_{||}^2 v_e^2}. \tag{4.20}$$

Both pitch angle and energy scatter matter in the intense field limit. Importantly, the intense field limit depends on $\tilde{E}_{||}$, and is smaller by roughly

$$1 \gg \frac{\left(\Delta v_{||}\right)_{\mathrm{is}}^2}{v_e^2} \sim \left|\frac{e\tilde{E}_{||}}{mk_{||}v_e^2}\right|, \tag{4.21}$$

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when $(\Delta v_{||})^2_{\text{is}} \gg (\Delta v_{||})^2_{\nu}$. Consequently, intense field LHCD is expected to be less efficient than QL predictions indicate. The result here, namely (4.19), as well as plasma edge turbulence likely explain some of the decrease in efficiency observed in experiments (Bonoli 2014). The nonlinear reduction by $(\Delta v_{||})^2_{\text{is}}/v_e^2 \ll 1$ occurs because the bound electrons are unable to carry current. The normalised efficiency is reduced, even though less power is absorbed in the narrow collisional boundary layers, because the bound region containing the non-current carrying electrons becomes much wider as the applied lower hybrid wave amplitude becomes larger. Additional island structures are expected to result in further reductions.

5. Discussion

It is normally assumed that a Maxwellian unperturbed distribution evolves due to QL effects generating non-Maxwellian features. However, the argument in § 1 indicates that these non-Maxwellian features should be taken as indication that a QL treatment is no longer valid and velocity space structure associated with island formation is starting to enter. As a result, the nonlinear term in the perturbed kinetic equation is no longer negligible and a full nonlinear solution is required. Interestingly, the unperturbed distribution remains nearly Maxwellian even in the presence of island formation and when the applied RF becomes intense as long as (4.21) is satisfied. In the intense limit collisional boundary layers form in very narrow regions about the separatrix and resolve the step function behaviour between the bound region and the two circulating regions. As only the circulating electrons can carry current and the island width is much larger than the collisional boundary layer width, less current is driven as indicated by comparing the intense field efficiency of (4.19) with the usual QL current drive efficiency of (4.20). This reduction in the efficiency offers at least a partial explanation of the experimentally observed decrease in current drive efficiency.

Note added in proofs. François Waelbroeck kindly brought to my attention a classic paper by Zakharov and Karpman (1963) in which they demonstrate that Landau damping is a collisional process. The treatment in Catto (2025b) verifies their results by solving the steady state driven plasma wave problem linearly and nonlinearly with collisions. Details associated with the collision operator and the coefficient of the power absorbed in the weakly collisional, large plasma wave amplitude limit differ, but the procedure is broadly the same. Their pioneering treatment should be consulted for full details. Sugihara *et al.* (1981) inappropriately modify the solution of Zakharov and Karpman (1963) to remove the step function behavior at the collisional boundary layer enclosing the separatrix to obtain a piecewise continuous solution for the distribution function that no longer properly matches to the non-resonant electron distribution function (Catto 2025a). Their flawed result for the rf power absorbed agrees with the collisionless result of Canobbio (1972) as they note. The work here agrees with Zakharov and Karpman (1963) within numerical coefficients associated with somewhat different forms for the collision operator.

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Declaration of interests

The author reports no conflict of interest.

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