Adv. Appl. Probab. 1–38 (2025)
*doi:*10.1017/apr.2025.10043

BRANCHING PARTICLE SYSTEMS WITH MUTUALLY CATALYTIC INTERACTIONS

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Abstract

We study a continuous-time mutually catalytic branching model on the $\mathbb{Z}^d$. The model describes the behavior of two different populations of particles, performing random walk on the lattice in the presence of branching, that is, each particle dies at a certain rate and is replaced by a random number of offspring. The branching rate of a particle in one population is proportional to the number of particles of another population at the same site. We study the long time behavior for this model, in particular, coexistence and noncoexistence of two populations in the long run. Finally, we construct a sequence of renormalized processes and use duality techniques to investigate its limiting behavior.

Keywords: Mutually catalytic branching; coexistence; finite system scheme

2020 Mathematics Subject Classification: Primary 60J80
Secondary 60J85

1. Introduction

1.1. Background and motivation

In the last four decades there has been a lot of interest in spatial branching models. These models include branching random walks, branching Brownian motion, superprocesses, and so on. During the last three decades, branching models with interactions were studied very extensively on the level of continuous state models and particle models. We provide here a partial list of branching models with interactions that were studied in the literature.

Models with catalytic branching, where one population catalyzes another were studied in [13, 14, 30]. For measure-valued diffusions with mutual catalytic branching, see [11, 12, 15, 16]. Models with symbiotic branching (these are models with a correlation in branching laws of two populations) were investigated in [1, 3–5, 20, 22, 23], among others. Infinite rate branching models was introduced in [26, 29] and studied later in [17–19, 27, 28]. In addition, various particle models were introduced in [2, 25].

Received 2 November 2025; accepted 2 November 2025.
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Let us say a few words about mutually catalytic branching model in the continuous state setting introduced in [16].

Dawson and Perkins [16] constructed the model with $\mathbb{Z}^d$ being a space of sites and $(u, v) \in \mathbb{R}_+^{\mathbb{Z}^d} \times \mathbb{R}_+^{\mathbb{Z}^d}$ a pair which undergo random migration and continuous state mutually catalytic branching. The random migration is described by a $\mathbb{Z}^d$-valued Markov chain with the associated Q-matrix, $Q = (q_{ij})$, subject to certain technical conditions on Q and associated transition probabilities (see [16, p. 1090] and, in particular, (H_0), (H_1), and (H_2) there). The branching rate of one population at a site is proportional to the mass of the other population at the site. The system is modeled by the following infinite system of stochastic differential equations (SDEs):

$$\begin{cases} u_t(x) = u_0(x) + \int_0^t u_s Q(x)\,\mathrm{d}s + \int_0^t \sqrt{\widetilde{\gamma} u_s(x) v_s(x)}\,\mathrm{d}B_s^x, & t \geq 0,\ x \in \mathbb{Z}^d, \\ v_t(x) = v_0(x) + \int_0^t v_s Q(x)\,\mathrm{d}s + \int_0^t \sqrt{\widetilde{\gamma} u_s(x) v_s(x)}\,\mathrm{d}W_s^x, & t \geq 0,\ x \in \mathbb{Z}^d, \end{cases} \tag{1.1}$$

where $\{B_s^x\}_{x\in\mathbb{Z}^d}$, $\{W_s^x\}_{x\in\mathbb{Z}^d}$ are collections of one-dimensional independent Brownian motions, and $\tilde{\gamma} > 0$.

One of the main questions which was introduced in [16] is the question of coexistence and noncoexistence of types in the long run. In particular, it has been proved that there is a clear dichotomy: coexistence is possible if the migration is transient, that is in dimensions $d \geq 3$, and is impossible if the migration is recurrent, that is if $d \leq 2$.

The above model is a particular case of so-called 'interacting mutually catalytic diffusions' studied by many authors in different settings, see for example, [8, 11, 33].

Cox et al. [6] analyzed the behavior of the Dawson–Perkins system with very large but finite space of sites in comparison with the corresponding model with infinite space of sites. This type of question arises if, for example, one is interested in determining what simulations of finite systems can say about the corresponding infinite spatial system. In [6], the authors considered a sequence of finite subsets of $\mathbb{Z}^d$ increasing to the whole $\mathbb{Z}^d$ and check the limiting behavior of mutually catalytic models restricted to these sets, while time is also suitably rescaled. It is called a 'finite system scheme'. This concept appeared in [7, 9, 10].

To formulate a result from [6] we need to introduce the following construction. Fix $n \in \mathbb{N}$. Define $\Lambda_n = [-n, n]^d \cap \mathbb{Z}^d$. Let $Q = (q(i, j))_{i,j\in\mathbb{Z}^d}$ be the Q-matrix of $\mathbb{Z}^d$-valued Markov chain. Define $Q^n = (q^n(i, j))_{i,j\in\Lambda_n}$ as follows

$$q^n(0, y - x) = \sum_{z \in I(y-x)} q(0, z), \tag{1.2}$$

where

$$I(x) = \left\{ y \in \mathbb{Z}^d \mid y = x\,(\mathrm{mod}\,()\Lambda_n) \right\}.$$

Consider a process that solves Dawson–Perkins equations (1.1) with state space being the torus Λ_n, i.e.

$$\begin{cases} u_t^n(x) = u_0^n(x) + \int_0^t u_s^n Q^n(x)\,\mathrm{d}s + \int_0^t \sqrt{\widetilde{\gamma} u_s^n(x) v_s^n(x)}\,\mathrm{d}B_s^x, & t \geq 0,\ x \in \Lambda_n, \\ v_t^n(x) = v_0^n(x) + \int_0^t v_s^n Q^n(x)\,\mathrm{d}s + \int_0^t \sqrt{\widetilde{\gamma} u_s^n(x) v_s^n(x)}\,\mathrm{d}W_s^x, & t \geq 0,\ x \in \Lambda_n, \end{cases} \tag{1.3}$$

where $\{B_s^x\}_{x\in\Lambda_n}$, $\{W_s^x\}_{x\in\Lambda_n}$ are collections of one-dimensional independent Brownian motions, and $\tilde{\gamma} > 0$.

We denote such a process by $(U^n_t, V^n_t)_{t\geq 0} := ((u^n_t(x))_{x\in\Lambda_n}, (v^n_t(x))_{x\in\Lambda_n})_{t\geq 0}$. Define time scaling depending on the system size

$$\beta_n(t) = |\Lambda_n|\, t = (2n+1)^d t.$$

Define

$$\mathbf{U}^n_t = \sum_{x\in\Lambda_n} u^n_t(x), \qquad \mathbf{V}^n_t = \sum_{x\in\Lambda_n} v^n_t(x), \tag{1.4}$$

and renormalized process

$$D_n((U^n_\cdot, V^n_\cdot)) = \left(D^1_n, D^2_n\right) = \frac{1}{|\Lambda_n|}\left(\sum_{x\in\Lambda_n} u^n_\cdot(x), \sum_{x\in\Lambda_n} v^n_\cdot(x)\right). \tag{1.5}$$

In what follows $\mathcal{L}(\cdot)$ denotes the law of random variable or process.

It was shown in [6] that in dimensions $d \geq 3$ the sequence D_n-processes with suitably rescaled time is tight and converges to a diffusion.

Theorem 1.1. (Theorem 1(a) in [6]). *Let $d \geq 3$, and let Q be a generator of a simple random walk on $\mathbb{Z}^d$. Assume that*

$$u^n_0(x) = \theta_1,\, v^n_0(x) = \theta_2, \;\; \forall x \in \Lambda_n.$$

Then

$$\mathcal{L}\big(D_n\big(U^n_{\beta_n(t)}, V^n_{\beta_n(t)}\big)_{t\geq 0}\big) \longrightarrow \mathcal{L}\left((X_t, Y_t)_{t\geq 0}\right), \quad \text{as } n\to\infty,$$

where $(X_t, Y_t)_{t\geq 0}$ is the unique weak solution for the following system of SDEs:

$$\begin{cases} \mathrm{d}X_t = \sqrt{\tilde{\gamma} X_t Y_t}\,\mathrm{d}w^1(t), & t\geq 0,\\ \mathrm{d}Y_t = \sqrt{\tilde{\gamma} X_t Y_t}\,\mathrm{d}w^2(t), & t\geq 0, \end{cases} \tag{1.6}$$

with initial conditions $(X_0, Y_0) = \bar{\theta} = (\theta_1, \theta_2)$, where w^1, w^2 are two independent standard Brownian motions.

In this paper we consider the Dawson–Perkins mutually catalytic model for *particle systems* and study its properties. As for other particle models in the presence of interactions, they have been considered earlier by many authors. A partial list of examples follows.

Birkner [2] studied a system of particles performing random walks on $\mathbb{Z}^d$ and performing branching; the rate of branching of any particle on a site depends on the number of other particles at the same site (this is the 'catalytic' effect). Birkner introduced a formal construction of such processes, via solutions of certain stochastic equations, proved existence and uniqueness theorems for these equations, and studied the properties of the processes. Under suitable assumptions he proved the existence of an equilibrium distribution for shift-invariant initial conditions. He also studied survival and extinction of the process in the long run. Note that the construction of the process in [2] was motivated by the construction of Ligget and Spitzer [31].

Among many other works where branching particle systems with catalysts were studied, we can mention [25] and [30]. For example, Kesten and Sidoravicius [25] investigated the survival/extinction of two particle populations A and B. Both populations perform an independent random walk. The B particles perform a branching random walk, but with a birth rate of new particles which is proportional to the number of A particles which coincide with the appropriate B particles. It was shown that for a choice of parameters the system becomes (locally) extinct in all dimensions.

In [30] catalytic discrete state branching processes with immigration were defined as strong solutions of stochastic integral equations. Li and Ma [30] proved limit theorems for these processes.

In this paper we consider two interactive populations: to be more precise, we construct the so-called mutually catalytic branching model and study its long time behavior and finite systems scheme.

1.2. Paper overview

In the next two subsections we introduce our model and state main results. In Section 2 the process is formally constructed and main results are stated. Sections 3–6 are devoted to the proofs of our results.

2. Our Model and Main Results

2.1. Description of the model

Let us define the following interactive particle system. We consider two populations on a countable set of sites $S \subset \mathbb{Z}^d$, where particles in both populations move as independent Markov chains on S with rate of jumps $\kappa > 0$, and symmetric transition jump probabilities

$$p_{x,y} = p_{y,x}, \quad x, y \in S. \tag{2.1}$$

They also undergo branching events. In order to define our model formally, we are following the ideas of [2].

Let $\{\nu_k\}_{k\geq 0}$ be the branching law. Suppose that Z is a random variable distributed according to ν. We assume that branching law is critical and has a finite variance:

$$\mathrm{E}(Z) = \sum_{k\geq 0} k\nu_k = 1, \quad Var(Z) = \sum_{k\geq 0} (k-1)^2 \nu_k = \sigma^2 < \infty. \tag{2.2}$$

The pair of processes (ξ, η) describes the time evolution of the following 'particle' model. Between branching events in ξ and η populations move as independent Markov chains on S with rate of jumps κ and transition probabilities p_{xy}, $x, y \in S$. Fix some $\gamma > 0$. The 'infinitesimal' rate of a branching event for a particle from population ξ at site x at time t is equal to $\gamma\eta_t(x)$; similarly the 'infinitesimal' rate of a branching event for a particle from population η at site x at time t is equal to $\gamma\xi_t(x)$. When a 'branching event' occurs, a particle dies and is replaced by a random number of offspring, distributed according to the law $\{\nu_k\}_{k\geq 0}$, independently from the history of the process. To define a process formally, as a solution to a system of equations we need further notation. Note that construction of the process follows the steps in [2].

The Markov chain is defined in the following way. Let (W_t, P) be a continuous-time S-valued Markov chain with rate of jumps $\kappa > 0$, and symmetric transition jump probabilities $p_{x,y} = p_{y,x}$, $x, y \in S$. Set $p_t(x, y) = P(W_t = y | W_0 = x)$ as transitions probabilities. Let $Q = (q_{x,y})$ denote the associated Q-matrix; that is $q_{x,y} = \kappa p_{x,y}$ is the jump rate from x to y (for $x \neq y$) and

$q_{x,x} = -\sum_{y\neq x} q_{x,y} = -\kappa > -\infty$. Clearly, by our assumptions on transition jump probabilities, the Q-matrix is symmetric ($q_{x,y} = q_{y,x}$). Define the Green function for every $x, y \in S$:

$$g_t(x, y) = \int_0^t p_s(x, y)\, \mathrm{d}s. \tag{2.3}$$

Note that if our motion process is a symmetric random walk on S, hence, with certain abuse of notation, $g_t(x, y) = g_t(y, x) = g_t(x - y)$, in particular $g_t(x, x) = g_t(0)$.

Let $P_t f(x) = \sum_y p_t(x, y)f(y)$ be the semigroup associated with the Markov chain W and $Qf(x) = \sum_y q_{x,y} f(y)$ is its generator.

Remark 2.1. If W is a symmetric random walk, then clearly $g_\infty(0) < \infty$ means that W is transient, and $g_\infty(0) = \infty$ implies that W is recurrent.

Let $\mathcal{F} = (\mathcal{F}_t)_{t\geq 0}$ be a (right-continuous, complete) natural filtration. In what follows, when we call a process martingale, we mean that it is an $\mathcal{F}_t$-martingale.

Let

$$\{N^{\mathrm{RW}_\xi}_{x,y}\}_{x,y\in S, x\neq y}, \quad \{N^{\mathrm{RW}_\eta}_{x,y}\}_{x,y\in S, x\neq y}, \quad \{N^{\mathrm{br}_\xi}_{x,k}\}_{x\in S, k\in\mathbb{Z}_+}, \quad \{N^{\mathrm{br}_\eta}_{x,k}\}_{x\in S, k\in\mathbb{Z}_+}$$

denote independent Poisson point processes on $\mathbb{R}_+ \times \mathbb{R}_+$. We assume that, for any $x, y \in S$, $x \neq y$ both Poisson point processes $N^{\mathrm{RW}_\xi}_{x,y}$ and $N^{\mathrm{RW}_\eta}_{x,y}$ have intensity measure $\kappa p_{x,y}\, \mathrm{d}s \otimes \mathrm{d}u$. Similarly, we assume that, for any $x \in S$, $k \in \mathbb{Z}_+$ both Poisson point processes $N^{\mathrm{br}_\xi}_{x,k}$ and $N^{\mathrm{br}_\eta}_{x,k}$ have intensity measure $\nu_k\, \mathrm{d}s \otimes \mathrm{d}u$. We assume that the above Poisson processes are $\mathcal{F}$-adapted in the 'time' s-coordinate.

Now we are going to define the pair of processes $(\xi_t, \eta_t)_{t\geq 0}$ where $(\xi_t, \eta_t) \in \mathbb{N}_0^S \times \mathbb{N}_0^S$, and $\mathbb{N}_0$ denotes the set of non-negative integers.

For any $x \in S$, $\xi_t(x)$ counts the number of particles from the first population at site x at time t. Similarly, for any $x \in S$, $\eta_t(x)$ counts the number of particles from the second population at site x at time t.

Now we are ready to describe $(\xi_t, \eta_t)_{t\geq 0}$ formally as a solution of the following system of equations:

$$\begin{aligned}
\xi_t(x) &= \xi_0(x) + \sum_{y\neq x} \left\{ \int_0^t \int_{\mathbb{R}_+} 1_{\{\xi_{s-}(y)\geq u\}} N^{\mathrm{RW}_\xi}_{y,x}(\mathrm{d}s\, \mathrm{d}u) - \int_0^t \int_{\mathbb{R}_+} 1_{\{\xi_{s-}(x)\geq u\}} N^{\mathrm{RW}_\xi}_{x,y}(\mathrm{d}s\, \mathrm{d}u) \right\} \\
&\quad + \sum_{k\geq 0} \int_0^t \int_{\mathbb{R}_+} (k-1) 1_{\{\gamma\eta_{s-}(x)\xi_{s-}(x)\geq u\}} N^{\mathrm{br}_\xi}_{x,k}(\mathrm{d}s\, \mathrm{d}u), \quad t \geq 0,\ x \in S, \qquad (2.4) \\
\eta_t(x) &= \eta_0(x) + \sum_{y\neq x} \left\{ \int_0^t \int_{\mathbb{R}_+} 1_{\{\eta_{s-}(y)\geq u\}} N^{\mathrm{RW}_\eta}_{y,x}(\mathrm{d}s\, \mathrm{d}u) - \int_0^t \int_{\mathbb{R}_+} 1_{\{\eta_{s-}(x)\geq u\}} N^{\mathrm{RW}_\eta}_{x,y}(\mathrm{d}s\, \mathrm{d}u) \right\} \\
&\quad + \sum_{k\geq 0} \int_0^t \int_{\mathbb{R}_+} (k-1) 1_{\{\gamma\xi_{s-}(x)\eta_{s-}(x)\geq u\}} N^{\mathrm{br}_\eta}_{x,k}(\mathrm{d}s\, \mathrm{d}u), \quad t \geq 0,\ x \in S.
\end{aligned}$$

Why do these equations actually describe our processes? The first sum on the right-hand side of equations for ξ and η describes the random walks of particles, and the second sum describes their branching. The first integrals in the first sums describe all particles jumping to site x from different sites $y \neq x$. The second integrals in the first sum describe particles that leave site x. The last integral describes the death of a particle at site x and the birth of its k offspring, so after that event the number of particles at the site has changed by $k-1$. The branching events at site x happen with the infinitesimal rate proportional to the product of the number of particles of both populations at site x.

Definition 2.1. The process (ξ_t, η_t) solving (2.4) is called a mutually catalytic branching process with initial conditions (ξ_0, η_0).

2.2. Main results

We start with stating the result on the existence and uniqueness of the solution for the system of equations (2.4). This implies that the process we described in the introduction does exist and is defined uniquely via the solution to (2.4). In the next theorem, we formulate the result for finite initial conditions, i.e. each population has a finite number of particles at initial time ($t = 0$). First, we introduce another piece of notation. For $m \in \mathbb{N}$, define the L^m-norm of $\varphi \in \mathbb{Z}^S$:

$$\|\varphi\|_m := \left(\sum_{i \in S} |\varphi(i)|^m \right)^{1/m}. \tag{2.5}$$

Similarly, for any $(\varphi, \psi) \in \mathbb{Z}^S \times \mathbb{Z}^S$, $(\varphi, \psi, \tilde{\varphi}, \tilde{\psi}) \in \left(\mathbb{Z}^S\right)^4$, with some abuse of notation, we define

$$\|(\varphi, \psi)\|_m := \left(\sum_{i \in S} \left(|\varphi(i)|^m + |\psi(i)|^m \right) \right)^{1/m}, \tag{2.6}$$

$$\|(\varphi, \psi, \tilde{\varphi}, \tilde{\psi})\|_m := \left(\sum_{i \in S} \left(|\varphi(i)|^m + |\psi(i)|^m + |\tilde{\varphi}(i)|^m + |\tilde{\psi}(i)|^m \right) \right)^{1/m}.$$

In addition let us define the space of functions E_{fin}:

$$E_{\text{fin}} = \left\{ f : S \to \mathbb{N}_0 \mid \|f\|_1 < \infty \right\}.$$

We equip E_{fin} with the metric: $d_{E_{\text{fin}}}(f, g) = \|f - g\|_1$ for any $f, g \in E_{\text{fin}}$.

Theorem 2.1. *Let $S \subset \mathbb{Z}^d$.*

(a) *For any initial conditions $(\xi_0, \eta_0) \in E_{\text{fin}} \times E_{\text{fin}}$ there is a unique strong solution $(\xi_t, \eta_t)_{t \geq 0}$ to (2.4), taking values in $E_{\text{fin}} \times E_{\text{fin}}$.*

(b) *The solution $\{(\xi_t, \eta_t),\ t \geq 0\}$ to (2.4) is a Markov process.*

It is possible to generalize the result to some infinite mass initial conditions case but since this is not the goal of this paper it will be done elsewhere.

Let (ξ, η) be the process constructed in Theorem 2.1 with finite initial conditions. Denote

$$\boldsymbol{\xi}_t = \sum_{i \in S} \xi_t(i), \quad \boldsymbol{\eta}_t = \sum_{i \in S} \eta_t(i), \quad t \geq 0.$$

That is, $\boldsymbol{\xi}$ is the total mass process of ξ, and $\boldsymbol{\eta}$ is the total mass process of η. Clearly, by construction, $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ are non-negative local martingales and, hence, by the martingale convergence theorem there exist almost surely (a.s.) limits

$$\boldsymbol{\xi}_\infty = \lim_{t \to \infty} \boldsymbol{\xi}_t, \quad \boldsymbol{\eta}_\infty = \lim_{t \to \infty} \boldsymbol{\eta}_t.$$

Now we are ready to give a definition of coexistence or noncoexistence.

Definition 2.2. Let (ξ, η) be a unique strong solution to (2.4) with $(\xi_0, \eta_0) \in E_{\text{fin}} \times E_{\text{fin}}$. We say that coexistence is possible for (ξ, η) if $\mathrm{P}(\boldsymbol{\xi}_\infty \boldsymbol{\eta}_\infty > 0) > 0$. We say that coexistence is impossible for (ξ, η) if $\mathrm{P}(\boldsymbol{\xi}_\infty \boldsymbol{\eta}_\infty > 0) = 0$.

Convention We say that the motion process for the *mutually catalytic branching process* on $S = \mathbb{Z}^d$, is the nearest-neighbor random walk if

$$p_{x,y} = \frac{1}{2d} \quad \text{for } y = x \pm e_i,$$

for e_i a unit vector in an axis direction, $i = 1, \ldots, d$.

We prove that in the finite initial conditions case, with motion process being the nearest-neighbor random walk, the coexistence is possible if and only if the random walk is transient. Recall that the nearest-neighbor random walk is recurrent in dimensions $d = 1, 2$, and it is transient in dimensions $d \geq 3$. Then we have the following theorem.

Theorem 2.2. *Let $S = \mathbb{Z}^d$ and assume that the motion process is the nearest-neighbor random walk. Let $(\xi_0, \eta_0) \in E_{\text{fin}} \times E_{\text{fin}}$ with $\boldsymbol{\xi}_0 \boldsymbol{\eta}_0 > 0$.*

(a) *If $d \geq 3$, then coexistence of types is possible.*

(b) *If $d \leq 2$, then coexistence of types is impossible.*

The proof is simple and based on the following observation: if there is a finite number of particles and the motion is recurrent, the particles will meet an infinite number of times, and eventually one of the populations dies out, due to the criticality of the branching mechanism. On the other hand, if the motion is transient, there exists a finite time such that after this time the particles of different populations never meet, and hence there is a positive probability of survival of both populations.

Finally we are interested in a finite system scheme. We construct a system of renormalized processes started from an exhausting sequence of finite subsets of $\mathbb{Z}^d$, $\Lambda_n \subset \mathbb{Z}^d$. The duality techniques will be used to investigate its limiting behavior.

Define

$$\Lambda_n = \left\{ x \in \mathbb{Z}^d \mid \forall i = 1, \ldots, d, \, |x_i| \leq n \right\} \subseteq \mathbb{Z}^d,$$
$$|\Lambda_n| = (2n+1)^d.$$

Convention Let $S = \Lambda_n$. We say that the motion process is the nearest-neighbor random walk on Λ_n if its transition jump probabilities are given by

$$p^n_{x,y} = p^n_{0,y-x} = \begin{cases} \frac{1}{2d}, & \text{if } |x-y| = 1, \\ 0, & \text{otherwise,} \end{cases}$$

where '$y - x$' is the difference on the torus Λ_n.

Fix $\bar{\theta} = (\theta_1, \theta_2)$ with $\theta_1, \theta_2 \in \mathbb{N}_0$. Assume $\bar{\boldsymbol{\theta}} = (\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)$, where $\boldsymbol{\theta}_i = (\theta_i, \theta_i, \ldots) \in \mathbb{N}_0^{\Lambda_n}$, $i = 1, 2$. Let $(\xi_t, \eta_t)_{t \geq 0}$ be the mutually catalytic branching process with initial conditions $(\xi_0, \eta_0) = \bar{\boldsymbol{\theta}}$, and site space $S = \Lambda_n$, and motion process being the nearest-neighbor walk on Λ_n.

Set

$$\boldsymbol{\xi}_t^n = \sum_{j\in\Lambda_n} \xi_j(t), \qquad \boldsymbol{\eta}_t^n = \sum_{j\in\Lambda_n} \eta_j(t). \tag{2.7}$$

We define the following time change:

$$\beta_n(t) = |\Lambda_n|\, t, \qquad t \geq 0.$$

Our goal is to identify the limiting distribution of

$$\frac{1}{|\Lambda_n|}\left(\boldsymbol{\xi}^n_{\beta_n(t)}, \boldsymbol{\eta}^n_{\beta_n(t)}\right),$$

as $n \to \infty$, for all $t \geq 0$.

Theorem 2.3. *Let $d \geq 3$, and assume that*

$$\gamma\sigma^2 < \frac{1}{\sqrt{3^5}\left(\frac{1}{2}g_\infty(0) + 1\right)}, \tag{2.8}$$

and $\sum_k k^3 \nu_k < \infty$. Then for any $T \in (0, 1]$, we have

$$\mathcal{L}\left(\frac{1}{|\Lambda_n|}\left(\boldsymbol{\xi}^n_{\beta_n(T)}, \boldsymbol{\eta}^n_{\beta_n(T)}\right)\right) \longrightarrow \mathcal{L}\left(X_T, Y_T\right), \quad \text{as } n \to \infty,$$

where $(X_t, Y_t)_{t\geq 0}$ is a solution of the following system of SDEs:

$$\begin{cases} \mathrm{d}X_t = \sqrt{\gamma\sigma^2 X_t Y_t}\, \mathrm{d}w^1(t), & t \geq 0, \\ \mathrm{d}Y_t = \sqrt{\gamma\sigma^2 X_t Y_t}\, \mathrm{d}w^2(t), & t \geq 0, \end{cases} \tag{2.9}$$

with initial conditions $(X_0, Y_0) = \bar{\theta}$, where w^1, w^2 are two independent standard Brownian motions.

Remark 2.2. The above theorem gives convergence of one-dimensional distributions of the rescaled processes $(\boldsymbol{\xi}^n, \boldsymbol{\eta}^n)$ to the one-dimensional distributions of the solution of (2.9) starting at initial conditions $\bar{\theta} \in \mathbb{N}_0^2$. It seems possible to treat a more general class of initial conditions, for example, independent and identically distributed (i.i.d.) configurations on Λ_n with mean vector $\bar{\theta} = (\theta_1, \theta_2) \in \mathbb{R}_+^2$. However, this will make the argument more technically involved, thus we decided to treat this case elsewhere.

Remark 2.3. The condition $\gamma\sigma^2 < \frac{1}{\sqrt{\cdot 3^5}(\frac{1}{2}g_\infty(0)+1)}$ arises from the method of proof, which requires boundedness of the fourth moment of the dual processes (see Lemma 6.8, where this condition is applied). We conjecture, however, that the result holds without the additional constraint on $\gamma\sigma^2$, as is the case in the finite scheme for Dawson–Perkins processes.

The above result is similar, although a bit weaker, to the result in Theorem 1 in [6], where a finite scheme for the system of continuous SDEs is studied. The proof of Theorem 2.3 is based on the duality principle for our particle system and the result for SDEs in [6]. In fact, let us mention that the self-duality property for our mutually catalytic branching particle model (the property which is well known for processes solving equations of type (1.3)) does not hold. Thus, we use the so-called approximating duality technique to prove Theorem 2.3. The approximating duality technique was used in the past to resolve a number of weak uniqueness problems (see e.g. [32, 34]). We believe that using approximate duality to prove limit theorems is novel and this technique is of independent interest.

Let us note that it would be very interesting to extend the above results. First, it would be nice to address the question of coexistence/noncoexistence for more general motions and infinite mass initial conditions. As for extending results in Theorem 2.3, we would be interested to check what happens in the case of large γ, to prove 'functional convergence' result as in [6], and investigate the system's behavior for the case of recurrent motion, that is, in the dimensions $d = 1, 2$. We plan to address these problems in the future.

3. Existence and Uniqueness: Proof of Theorem 2.1

This section is devoted to the proof of Theorem 2.1. Note that our proofs follow closely the argument of Birkner [2] with suitable adaptation to the two types of case.

For any metric space D with metric d, let Lip(D) denote a set of Lipschitz functions on D. We say that $f : D \to \mathbb{R}$ is in Lip(D) if and only if there exists a positive constant $C \in \mathbb{R}_+$ such that for any $\varphi, \psi \in D$, $|f(\varphi) - f(\psi)| \le Cd(\varphi, \psi)$. Let $L^{(2)}$ denote the following operator: for a measurable function $f: E_{\text{fin}} \times E_{\text{fin}} \to \mathbb{R}$, let

$$\begin{aligned} L^{(2)}f(\varphi, \psi) = \kappa \sum_{x,y\in S} \varphi(x)p_{xy}\big(f\big(\varphi^{x\to y}, \psi\big) - f(\varphi, \psi)\big) \\ + \kappa \sum_{x,y\in S} \psi(x)p_{xy}\big(f\big(\varphi, \psi^{x\to y}\big) - f(\varphi, \psi)\big) \\ + \sum_{x\in S} \gamma\varphi(x)\psi(x) \sum_{k\ge 0} \nu_k \left(f(\varphi + (k-1)\delta_x, \psi) - f(\varphi, \psi)\right) \\ + \sum_{x\in S} \gamma\varphi(x)\psi(x) \sum_{k\ge 0} \nu_k \left(f(\varphi, \psi + (k-1)\delta_x) - f(\varphi, \psi)\right), \end{aligned}$$

where $\varphi^{x\to y} = \varphi + \delta_y - \delta_x$, i.e. $\varphi^{x\to y}(x) = \varphi(x) - 1$, $\varphi^{x\to y}(y) = \varphi(y) + 1$ and $\varphi^{x\to y}(z) = \varphi(z)$ for all $z \in S$ and $z \ne x, y$.

Theorem 2.1 follows immediately from the next lemma.

Lemma 3.1. *We have the following.*

(a) *For any initial conditions* $(\xi_0, \eta_0) \in E_{\text{fin}} \times E_{\text{fin}}$ *there is a unique strong solution* $(\xi_t, \eta_t)_{t\ge 0}$ *to* (2.4)*, taking values in* $E_{\text{fin}} \times E_{\text{fin}}$.

(b) *The solution* $\{(\xi_t, \eta_t),\ t \ge 0\}$ *to* (2.4) *is a Markov process.*

(c) *Let* $m \in \mathbb{N}$. *If* $\sum_k k^m \nu_k < \infty$, *there exists a constant* C_m *such that*

$$\mathrm{E}\left[\|(\xi_t, \eta_t)\|_m^m\right] \le \exp\left(C_m t\right) \|(\xi_0, \eta_0)\|_m^m . \tag{3.1}$$

(d) *For* $f \in \mathrm{Lip}(E_{\text{fin}} \times E_{\text{fin}})$, $\quad (\xi_0, \eta_0) \in E_{\text{fin}} \times E_{\text{fin}}$,

$$M^f(t) := f(\xi_t, \eta_t) - f(\xi_0, \eta_0) - \int_0^t L^{(2)}f(\xi_s, \eta_s)\,\mathrm{d}s, \tag{3.2}$$

is a martingale. Moreover, if $f \in \mathrm{Lip}(\mathbb{R}_+ \times E_{\text{fin}} \times E_{\text{fin}})$ *and there is constant* C^* *such that*

$$\left| \frac{\partial}{\partial s} f(s, \varphi, \psi) \right| \le C^* \, \|(\varphi, \psi)\|_2^2 \tag{3.3}$$

for any $\varphi, \psi \in E_{\text{fin}}$*, then*

$$N^f(t) := f(t, \xi_t, \eta_t) - f(0, \xi_0, \eta_0) - \int_0^t \left[L^{(2)} f(s, \xi_s, \eta_s) + \frac{\partial}{\partial s} f(s, \xi_s, \eta_s) \right] \mathrm{d}s, \tag{3.4}$$

is also a martingale.

Proof. Note that in our proof we follow ideas of the proof of Lemma 1 in Birkner [2].

(a) We have the collection of independent Poisson point processes $\{N_{x,y}^{\mathrm{RW}_\xi}\}_{x,y \in S}$, $\{N_{x,y}^{\mathrm{RW}_\eta}\}_{x,y \in S}$, $\{N_{x,k}^{\mathrm{br}_\xi}\}_{x \in S, k \in \mathbb{Z}_+}$, $\{N_{x,k}^{\mathrm{br}_\eta}\}_{x \in S, k \in \mathbb{Z}_+}$. Therefore, with probability 1 there is no more than one jump simultaneously. Then we can define a stopping time T_1: the first time a jump happens. The stopping time T_1 is strictly positive a.s., since $(\xi_0, \eta_0) \in E_{\text{fin}} \times E_{\text{fin}}$, and therefore the indicators $1_{\{\xi_{s-}(y) \ge u\}}$, $1_{\{\eta_{s-}(y) \ge u\}}$, $1_{\{\gamma \xi_{s-}(x) \eta_{s-}(x) \ge u\}}$ make the rate of the first jump finite. Then $(\xi_t, \eta_t) = (\xi_0, \eta_0)$ for $t \in [0, T_1)$. In the same way, we define a sequence of stopping times: $0 = T_0 < T_1 < T_2 < \cdots < \infty$, and again the rate of jumps is finite since for any $i \ge 1$, $(\xi_{T_i}, \eta_{T_i}) \in E_{\text{fin}} \times E_{\text{fin}}$ a.s., and this makes the rate of the $(i+1)$th jump a.s. finite by the same reasoning as for the first jump. Clearly, by construction, the process (ξ_t, η_t) is constant on the intervals $[T_i, T_{i+1})$. In order to show that this construction defines the process properly (that is, it does not explode in finite time), it is enough to show that $\lim_{n \to \infty} T_n = \infty$, a.s. To this end define $M_n = \sum_{x \in S} \left(\xi_{T_n}(x) + \eta_{T_n}(x) \right)$. Here M_n denotes the total number of particles in both populations at time T_n. Since the branching mechanism is critical (see (2.2)), it is easy to see that $\{M_n\}_{n \ge 0}$ is a non-negative martingale.

Indeed, suppose that T_n is the stopping time originated from a 'random walk' jump, that is, from a jump of one of the processes

$$R_t^{\xi,x} = \sum_{y \ne x} \left\{ \int_0^t \int_{\mathbb{R}_+} 1_{\{\xi_{s-}(y) \ge u\}} N_{y,x}^{\mathrm{RW}_\xi}(\mathrm{d}s\, \mathrm{d}u) - \int_0^t \int_{\mathbb{R}_+} 1_{\{\xi_{s-}(x) \ge u\}} N_{x,y}^{\mathrm{RW}_\xi}(\mathrm{d}s\, \mathrm{d}u) \right\}, \quad t \ge 0, \ x \in S,$$

or

$$R_t^{\eta,x} = \sum_{y \ne x} \left\{ \int_0^t \int_{\mathbb{R}_+} 1_{\{\eta_{s-}(y) \ge u\}} N_{y,x}^{\mathrm{RW}_\eta}(\mathrm{d}s\, \mathrm{d}u) - \int_0^t \int_{\mathbb{R}_+} 1_{\{\eta_{s-}(x) \ge u\}} N_{x,y}^{\mathrm{RW}_\eta}(\mathrm{d}s\, \mathrm{d}u) \right\}, \quad t \ge 0, \ x \in S.$$

In that case the total number of particles does not change (this can also be readily seen from our (2.4) since $\sum_{x \in S} R_t^{\xi,x} = \sum_{x \in S} R_t^{\eta,x} = 0$) and thus we have $M_n = M_{n-1}$. Alternatively T_n can be originated from the 'branching', that is from the jump of one of the processes $B_t^\xi = \sum_{x \in S} \sum_{k \ge 0} \int_0^t \int_{\mathbb{R}_+} (k-1) 1_{\{\gamma \eta_{s-}(x) \xi_{s-}(x) \ge u\}} N_{x,k}^{\mathrm{br}_\xi}(\mathrm{d}s\, \mathrm{d}u)$ or $B_t^\eta = \sum_{x \in S} \sum_{k \ge 0} \int_0^t \int_{\mathbb{R}_+} (k-1) 1_{\{\gamma \eta_{s-}(x) \xi_{s-}(x) \ge u\}} N_{x,k}^{\mathrm{br}_\eta}(\mathrm{d}s\, \mathrm{d}u)$. In this case one can easily get that

$$\mathrm{E}\left(M_n \mid M_0, \ldots, M_{n-1}\right) = M_{n-1} + \mathrm{E}\left(Z - 1\right) = M_{n-1},$$

where Z is distributed according to the branching law ν.

Therefore, by the well-known martingale convergence theorem $\sup_{n\geq 1} M_n < \infty$ a.s. [24, Theorem 1.6.4]. This implies that $\sup_n T_n = \infty$ a.s.

Now let us turn to the proof of uniqueness. Let $(\tilde{\xi}, \tilde{\eta})_t$ be another solution to (2.4) starting from the same initial conditions $\tilde{\xi}_0 = \xi^0,\ \tilde{\eta}_0 = \eta^0$. We see from (2.4) that $\tilde{\xi}_t(x) = \xi_t(x),\ \tilde{\eta}_t(x) = \eta_t(x)$ for all $x \in S$ and $t \in [0, T_1)$, and also that $\tilde{\xi}_{T_1} = \xi_{T_1},\ \tilde{\eta}_{T_1} = \eta_{T_1}$. Then, by induction, (ξ, η) and $(\tilde{\xi}, \tilde{\eta})$ agree on $[T_n, T_{n+1})$ for all $n \in \mathbb{N}$.

(b) Poisson processes have independent and stationary increments. Therefore, by construction described in (a), we can immediately see that the distribution of (ξ_{t+h}, η_{t+h}), given $\mathcal{F}_t$, depends only on (ξ_t, η_t), and hence the process $(\xi_t, \eta_t)_{t\geq 0}$ is Markov.

(c) Now we show that $(\xi_t, \eta_t)_{t\geq 0}$ satisfies (3.1). Define a sequence of stopping times

$$T_n := \inf\left\{t \geq 0 : \textstyle\sum_{x\in S} (\xi_t(x) + \eta_t(x)) > n\right\},\ n \geq 1. \tag{3.5}$$

Choose arbitrarily $m \in \mathbb{N}$ such that

$$\sum_k k^m \nu_k < \infty.$$

For $\varphi, \psi \in E_{\text{fin}}$, define

$$h_m(\varphi, \psi) := \|(\varphi, \psi)\|_m^m.$$

Then, by an appropriate version of the Itô formula, we have that $\{M^{h_m}_{t\wedge T_n}\}_{t\geq 0}$ is a martingale. Let $\varphi, \psi \in E_{\text{fin}}$. Apply the $L^{(2)}$ operator on $h_m(\varphi, \psi)$ to get

$$\begin{aligned}
L^{(2)} h_m(\varphi, \psi) = {} & \kappa \sum_{x,y} \varphi(x) p_{xy} \left\{ \left((\varphi(y)+1)^m - \varphi(y)^m \right) \right. \\
& \left. + \left((\varphi(x)-1)^m - \varphi(x)^m \right) \right\} \\
& + \kappa \sum_{x,y} \psi(x) p_{xy} \left\{ \left((\psi(y)+1)^m - \psi(y)^m \right) \right. \\
& \left. + \left((\psi(x)-1)^m - \psi(x)^m \right) \right\} \\
& + \sum_x \gamma \varphi(x)\psi(x) \sum_{k\geq 0} \nu_k \left\{ (\varphi(x)+k-1)^m - \varphi(x)^m \right\} \\
& + \sum_x \gamma \varphi(x)\psi(x) \sum_{k\geq 0} \nu_k \left\{ (\psi(x)+k-1)^m - \psi(x)^m \right\} \\
= {} & \kappa \sum_x \varphi(x) \sum_{j=1}^m \binom{m}{j} (-1)^j \varphi(x)^{m-j} \\
& + \kappa \sum_{x,y} \varphi(x) p_{xy} \sum_{j=1}^m \binom{m}{j} \varphi(y)^{m-j} \\
& + \kappa \sum_x \psi(x) \sum_{j=1}^m \binom{m}{j} (-1)^j \psi(x)^{m-j}
\end{aligned}$$

$$
\begin{aligned}
&+\kappa\sum_{x,y}\psi(x)p_{xy}\sum_{j=1}^{m}\binom{m}{j}\psi(y)^{m-j}\\
&+\sum_{x}\gamma\varphi(x)\psi(x)\sum_{k\geq 0}\nu_k\sum_{j=2}^{m}\binom{m}{j}\varphi(x)^{m-j}(k-1)^j\\
&+\sum_{x}\gamma\varphi(x)\psi(x)\sum_{k\geq 0}\nu_k\sum_{j=2}^{m}\binom{m}{j}\psi(x)^{m-j}(k-1)^j.
\end{aligned}
$$

where in the last equality we used the binomial expansion, the fact that $\sum_y p_{xy}=1$ and our assumptions $\sum_{k\geq 0}(k-1)\nu_k=0$. Now we can estimate

$$
\begin{aligned}
\left|L^{(2)}h_m(\varphi,\psi)\right| \leq \kappa\left(\sum_{j=1}^{m}\binom{m}{j}\right)&\left(h_m(\varphi,\psi)+\sum_{x,y}p_{xy}\left[\varphi(x)^m+\varphi(y)^m+\psi(x)^m+\psi(y)^m\right]\right)\\
&+\left(3\gamma\sum_{j=2}^{m}\binom{m}{j}\sum_{k}\nu_k(k-1)^j\right)h_m(\varphi,\psi)
\end{aligned}
\tag{3.6}
$$

where we used the following simple inequalities: for $m\geq j\geq 1$,

$$
\varphi(x)^{m-j}\leq\varphi(x)^{m-1}
$$

(recall that $\varphi(x)$ is a non-negative integer number), and

$$
ab^{m-1}\leq a^m+b^m,\qquad \forall\, a,b\geq 0.
$$

Since, by our assumptions, $p_{xy}=p_{yx}$ for all $x,y\in S$, we have

$$
\sum_{x,y\in S}p_{xy}\left[\varphi(x)^m+\varphi(y)^m+\psi(x)^m+\psi(y)^m\right]\leq 4h_m(\varphi,\psi). \tag{3.7}
$$

Denote

$$
c_m:=\sum_{j=1}^{m}\binom{m}{j}=2^m-1,\qquad c'_m:=3\gamma\sum_{j=2}^{m}\binom{m}{j}\sum_{k\geq 0}\nu_k(k-1)^j<\infty.
$$

Then by (3.6) and (3.7) we get

$$
\left|L^{(2)}h_m(\varphi,\psi)\right|\leq(5\kappa c_m+c'_m)h_m(\varphi,\psi)\;=:C_m h_m(\varphi,\psi). \tag{3.8}
$$

Now recall that for any n, $\left\{M^{h_m}(t\wedge T_n)\right\}_{t\geq 0}$ is a martingale. Therefore,

$$
\begin{aligned}
\mathrm{E}\left[h_m\left(\xi_{t\wedge T_n},\eta_{t\wedge T_n}\right)\right]&=h_m(\xi^0,\eta^0)+\mathrm{E}\left[\int_0^{t\wedge T_n}L^{(2)}h_m(\xi_s,\eta_s)\,\mathrm{d}s\right]\\
&\leq h_m(\xi^0,\eta^0)+C_m\int_0^t\mathrm{E}\left[\mathbf{1}_{\{s\leq T_n\}}h_m(\xi_s,\eta_s)\right]\,\mathrm{d}s\\
&\leq h_m(\xi^0,\eta^0)+C_m\int_0^t\mathrm{E}\left[h_m(\xi_{s\wedge T_n},\eta_{s\wedge T_n})\right]\,\mathrm{d}s.
\end{aligned}
$$

Thus, from Gronwall's lemma we get that

$$\mathrm{E}\left[h_m(\xi_{t\wedge T_n}, \eta_{t\wedge T_n})\right] \le \exp\left(C_m t\right) h_m(\xi^0, \eta^0),$$

uniformly in n. It is easy to see from (a) that $T_n \to \infty$ as $n \to \infty$, a.s. Thus, inequality (3.1) follows from Fatou's lemma by letting $n \to \infty$.

(d) Let $f \in \mathrm{Lip}(E_{\mathrm{fin}} \times E_{\mathrm{fin}})$. We wish to show that M^f is indeed a martingale. In order to do that, first we show that for any such f there is a constant $C = C(\kappa, p, \sigma, \nu, f)$ such that

$$\left|L^{(2)} f(\varphi, \psi)\right| \le C \left\|(\varphi, \psi)\right\|_2^2 \quad \text{for all } \varphi, \psi \in E_{\mathrm{fin}}. \tag{3.9}$$

We decompose $L^{(2)} f(\varphi, \psi)$ into two parts corresponding to motion and branching mechanisms:

$$L^{(2)} f(\varphi, \psi) = L_{\mathrm{RW}} f(\varphi, \psi) + L_{\mathrm{br}} f(\varphi, \psi), \tag{3.10}$$

where

$$\begin{aligned} L_{\mathrm{RW}} f(\varphi, \psi) = \kappa & \sum_{x,y\in S} \varphi(x) p_{xy} \left(f\left(\varphi^{x\to y}, \psi\right) - f(\varphi, \psi)\right) \\ & + \kappa \sum_{x,y\in S} \psi(x) p_{xy} \left(f\left(\varphi, \psi^{x\to y}\right) - f(\varphi, \psi)\right), \end{aligned} \tag{3.11}$$

$$\begin{aligned} L_{\mathrm{br}} f(\varphi, \psi) = & \sum_{x\in S} \gamma \varphi(x)\psi(x) \sum_{k\ge 0} \nu_k \left(f(\varphi + (k-1)\delta_x, \psi) - f(\varphi, \psi)\right) \\ & + \sum_{x\in S} \gamma \varphi(x)\psi(x) \sum_{k\ge 0} \nu_k \left(f(\varphi, \psi + (k-1)\delta_x) - f(\varphi, \psi)\right). \end{aligned} \tag{3.12}$$

Using the Lipshitz property of f we obtain

$$\begin{aligned} |L_{\mathrm{RW}} f(\varphi, \psi)| = & \left| \sum_{x,y\in S} \varphi(x) p_{xy} \left(f\left(\varphi^{(x,y)}, \psi\right) - f(\varphi, \psi)\right) \right. \\ & \left. + \sum_{x,y\in S} \psi(x) p_{xy} \left(f\left(\varphi, \psi^{(x,y)}\right) - f(\varphi, \psi)\right) \right| \\ & \le C_f \sum_{x,y\in S} \varphi(x) p_{xy} \left\| \left(\varphi^{(x,y)}, \psi\right) - (\varphi, \psi) \right\|_1 \\ & + C_f \sum_{x,y\in S} \psi(x) p_{xy} \left\| \left(\varphi, \psi^{(x,y)}\right) - (\varphi, \psi) \right\|_1 \\ & \le 2C_f \sum_{x,y\in S} \varphi(x) p_{xy} + 2C_f \sum_{x,y\in S} \psi(x) p_{xy} \\ = & \, 2C_f \left\|(\varphi, \psi)\right\|_1 \le 2C_f \left\|(\varphi, \psi)\right\|_2^2, \end{aligned}$$

where in the last inequality we used $\|(\varphi, \psi)\|_1 \le \|(\varphi, \psi)\|_2^2$, which holds since the functions $\{\varphi(x)\}_{x\in S}$ and $\{\psi(x)\}_{x\in S}$ are integer valued. Turning to $L_{\rm br} f(\varphi, \psi)$ we get

$$\begin{aligned}|L_{\rm br} f(\varphi, \psi)| &= \Bigg| \sum_{x\in S} \gamma\varphi(x)\psi(x) \sum_{k\ge 0} \nu_k \Big(\big[f(\varphi+(k-1)\delta_x, \psi) - f(\varphi, \psi)\big] \\ &\qquad + \big[f(\varphi, \psi+(k-1)\delta_x) - f(\varphi, \psi)\big]\Big) \Bigg| \\ &\le C_f \sum_{x\in S} \gamma\varphi(x)\psi(x) \sum_{k\ge 0} \nu_k \left(2\, \|(k-1)\delta_x\|_1\right) \\ &= \left(2C_f \sum_{k\ge 0} \nu_k |k-1|\right) \sum_{x\in S} \gamma\varphi(x)\psi(x) \\ &\le 2\gamma C_f \sum_{k\ge 0} 2\nu_k |k-1| \cdot \|(\varphi, \psi)\|_2^2\,,\end{aligned}$$

where in the last inequality we used the fact that $\varphi(x)\psi(x) \le \varphi(x)^2 + \psi(x)^2$. Thus, (3.9) holds with

$$C := C_f\left(2 + 2\gamma \sum_{k\ge 0} \nu_k |k-1|\right).$$

Consider a bounded $f \in {\rm Lip}(E_{\rm fin} \times E_{\rm fin})$, then M^f is a local martingale, so we have for all $t, h \ge 0$

$$\mathrm{E}\big[M^f((t+h)\wedge T_n)\big|\mathcal{F}_t\big] = M^f\ ((t+h)\wedge T_n)\ ., \tag{3.13}$$

where T_n is defined in (3.5). The right-hand side converges to $M^f(t)$ a.s., as $n\to\infty$. Then

$$\mathrm{E}\left[\int_0^{(t+h)\wedge T_n} L^{(2)} f(\xi_s, \eta_s)\,{\rm d}s \,\middle|\, \mathcal{F}_t\right] \to \mathrm{E}\left[\int_0^{t+h} L^{(2)} f(\xi_s, \eta_s)\,{\rm d}s \,\middle|\, \mathcal{F}_t\right],\ \text{a.s,}$$

as $n\to\infty$. Here, we used again the dominated convergence, since by (3.9)

$$\int_0^{(t+h)\wedge T_n} L^{(2)} f(\xi_s, \eta_s)\,{\rm d}s \le C \int_0^{t+h} \|(\xi_s, \eta_s)\|_2^2\ {\rm d}s$$

and the expectation on the right-hand side of the above inequality is bounded due to (3.1) and finite initial conditions, for which clearly $\|(\xi_0, \eta_0)\|_2^2 < \infty$. Thus, M^f is indeed a martingale in the case of bounded $f \in {\rm Lip}(E_{\rm fin} \times E_{\rm fin})$.

Next, consider $f \in {\rm Lip}(E_{\rm fin} \times E_{\rm fin})$ which is non-negative, but not necessarily bounded. Define $f_n(\varphi, \psi) := f(\varphi, \psi) \wedge n$. Note that f_n is bounded and $f_n \in {\rm Lip}(E_{\rm fin} \times E_{\rm fin})$ with Lipshitz constant $C_{f_n} \le C_f$. As $n\to\infty$, we have

$$M^{f_n}(t) \to M^f(t),\ \text{a.s.,}$$
$$\mathrm{E}\left[f_n(\xi_{t+h}, \eta_{t+h}) \mid \mathcal{F}_t\right] \to \mathrm{E}\left[f(\xi_{t+h}, \eta_{t+h}) \mid \mathcal{F}_t\right],\ \text{a.s.,}$$

by monotone convergence. Observe that $\left|L^{(2)} f_n(\varphi, \eta)\right| \le C\,\|(\varphi, \eta)\|_2^2$ uniformly in n. We thus obtain

$$\mathrm{E}\left[\int_0^{t+h} L^{(2)} f_n(\xi_s, \eta_s)\,{\rm d}s \middle| \mathcal{F}_t\right] \to \mathrm{E}\left[\int_0^{t+h} L^{(2)} f(\xi_s, \eta_s)\,{\rm d}s \middle| \mathcal{F}_t\right]$$

a.s. as $n \to \infty$ by the dominated convergence theorem. Therefore, M^f is a martingale for non-negative Lipschitz f. For the general case, we use the decomposition of $f \in \mathrm{Lip}(E_{\mathrm{fin}} \times E_{\mathrm{fin}})$ as $f = f^+ - f^-$, where $f^+ := \max(f, 0)$ and $f^- := \max(-f, 0)$.

The same proof holds for N^f too, since $\frac{\partial}{\partial s} f$ is bounded by (3.3). □

4. Proof of Theorem 2.2.

The aim of this section is to prove Theorem 2.2

Let (ξ_t, η_t) be the mutually catalytic branching process described in Theorem 2.2, starting at (ξ_0, η_0) with with $\boldsymbol{\xi}_0 + \boldsymbol{\eta}_0 < \infty$. Recall that $\boldsymbol{\xi}_t, \ \boldsymbol{\eta}_t, \ t \geq 0$, denote the total size of each population at time t:

$$\boldsymbol{\xi}_t = \sum_{x \in \mathbb{Z}^d} \xi_t(x) \quad \text{and} \quad \boldsymbol{\eta}_t = \sum_{x \in \mathbb{Z}^d} \eta_t(x).$$

4.1. Proof of Theorem 2.2 (a): transient case

The proof is simple and we decided to avoid technical details. The observation is as follows: since the motion of particles is transient and the number of particles in the original populations is finite, there exists a.s. a finite time $\hat{T}$ such that, if one suppresses the branching, the initial particles of different populations never meet after time $\hat{T}$. On the other hand, due to the finiteness of the number of particles, the total branching rate in the system is finite and, thus, there is a positive probability for the event that in the original particle system there is no branching event until time $\hat{T}$. On this event, particles of different populations never meet after time $\hat{T}$ and therefore there is a positive probability of survival of both populations.

4.2. Proof of Theorem 2.2(b): recurrent case

We would like to show that

$$\boldsymbol{\xi}_\infty \boldsymbol{\eta}_\infty = 0, \qquad \mathrm{P} - \text{a.s.}$$

First, recall why $\lim_{t\to\infty} \boldsymbol{\xi}_t \boldsymbol{\eta}_t = \boldsymbol{\xi}_\infty \boldsymbol{\eta}_\infty$ exists. By Itô's formula it is easy to see that $\left\{\boldsymbol{\xi}_t \boldsymbol{\eta}_t\right\}_{t \geq 0}$ is a non-negative local martingale, that is a non-negative supermartingale. By the martingale convergence theorem non-negative supermartingales converge a.s. as time goes to infinity. Hence,

$$\lim_{t\to\infty} \boldsymbol{\xi}_t \boldsymbol{\eta}_t = \boldsymbol{\xi}_\infty \boldsymbol{\eta}_\infty, \qquad \mathrm{P} - \text{a.s.}$$

Also note that $\left\{\boldsymbol{\xi}_t \boldsymbol{\eta}_t\right\}_{t \geq 0}$ is an integer-valued supermartingale. Therefore, there exists a random time T_0 such that

$$\boldsymbol{\xi}_t \boldsymbol{\eta}_t = \boldsymbol{\xi}_\infty \boldsymbol{\eta}_\infty \quad \text{for all } t \geq T_0. \tag{4.1}$$

Now assume that $\boldsymbol{\xi}_\infty \boldsymbol{\eta}_\infty > 0$, that is $\boldsymbol{\xi}_t > 0$ and $\boldsymbol{\eta}_t > 0$ for $t \geq T_0$. Since the motion is recurrent, there is probability one for a 'meeting' of two populations after time T_0 at some site. Moreover, on the event $\left\{\boldsymbol{\xi}_\infty \boldsymbol{\eta}_\infty > 0\right\}$, by recurrence, after time T_0, two populations spend an infinite amount of time 'together'. Since the branching rate is at least $\gamma > 0$, when particles of two populations spend time 'together' on the same site, we immediately get that eventually a branching event will happen with probability one. However, this is a contradiction with (4.1). Therefore, $\boldsymbol{\xi}_t = 0$ or $\boldsymbol{\eta}_t = 0$ for all $t \geq T_0$, that is one of the populations becomes extinct, and coexistence is not possible.

5. Moment Computations for $S = \Lambda_n$

In this section, we derive some useful moment estimates for (ξ_t, η_t) solving (2.4) in the case of $S = \Lambda_n$ for arbitrary $n \geq 1$ (recall that Λ_n is the torus defined in Section 1.1). These estimates will be essential for proving Theorem 2.3 in Section 6.

To simplify the notation we suppress dependence on 'n'. Throughout the section the motion process for a mutually catalytic process (ξ_t, η_t) is the nearest-neighbor random walk on $S = \Lambda_n$. The transition semigroup (respectively the transition density $\{p_t(\cdot, \cdot)_{t\geq 0}\}$, Q-matrix) of the motion process will be denoted by $\{P_t\}_{t\geq 0}$. The motion process will be the nearest-neighbor random walk on S. For the definition of conditional quadratic variation [35, Chapter III]. For $\psi \in \mathbb{Z}^S$ and $\varphi \in \mathbb{R}^S$ define the inner product

$$\langle \psi, \varphi \rangle := \sum_{x\in S} \psi(x)\varphi(x),$$

whenever the sum is absolutely convergent.

Lemma 5.1. *Assume that $S = \Lambda_n$. Let $(\xi_0, \eta_0) \in E_{\text{fin}} \times E_{\text{fin}}$. If $\varphi : S \to \mathbb{R}_+$, then*

$$\langle \xi_t, \varphi \rangle = \langle \xi_0, P_t\varphi \rangle + N^{\xi}_t(t, \varphi), \quad \langle \eta_t, \varphi \rangle = \langle \eta_0, P_t\varphi \rangle + N^{\eta}_t(t, \varphi), \tag{5.1}$$

where

$$\begin{aligned} N^{\xi}_s(t, \varphi) = \sum_{x\in S} \Bigg(\sum_{y\neq x} \Bigg\{ \int_0^s \int_{\mathbb{R}_+} & P_{t-r}\varphi(x) 1_{\xi_{r-}(y)\geq u} N^{\mathrm{RW}_\xi}_{y,x}(\mathrm{d}r\, \mathrm{d}u) \\ & - \int_0^s \int_{\mathbb{R}_+} P_{t-r}\varphi(x) 1_{\xi_{r-}(x)\geq u} N^{\mathrm{RW}_\xi}_{x,y}(\mathrm{d}r\, \mathrm{d}u) \Bigg\} - \int_0^s \xi_r Q(x) P_{t-r}\varphi(x)\, \mathrm{d}r \Bigg) \\ & + \sum_{x\in S} \sum_{k\geq 0} (k-1) \int_0^s \int_{\mathbb{R}_+} P_{t-r}\varphi(x) 1_{\{\gamma\eta_{r-}(x)\xi_{r-}(x)\geq u\}} N^{\mathrm{br}_\xi}_{x,k}(\mathrm{d}r\, \mathrm{d}u), \quad s \leq t, \end{aligned} \tag{5.2}$$

and

$$\begin{aligned} N^{\eta}_s(t, \varphi) = \sum_{x\in S} \Bigg(\sum_{y\neq x} \Bigg\{ \int_0^s \int_{\mathbb{R}_+} & P_{t-r}\varphi(x) 1_{\eta_{r-}(y)\geq u} N^{\mathrm{RW}_\eta}_{y,x}(\mathrm{d}r\, \mathrm{d}u) \\ & - \int_0^s \int_{\mathbb{R}_+} P_{t-r}\varphi(x) 1_{\eta_{r-}(x)\geq u} N^{\mathrm{RW}_\eta}_{x,y}(\mathrm{d}r\, \mathrm{d}u) \Bigg\} - \int_0^s \eta_r Q(x) P_{t-r}\varphi(x)\, \mathrm{d}r \Bigg) \\ & + \sum_{x\in S} \sum_{k\geq 0} (k-1) \int_0^s \int_{\mathbb{R}_+} P_{t-r}\varphi(x) 1_{\{\gamma\eta_{r-}(x)\xi_{r-}(x)\geq u\}} N^{\mathrm{br}_\eta}_{x,k}(\mathrm{d}r\, \mathrm{d}u), \quad s \leq t \end{aligned}$$

are orthogonal square-integrable $\mathcal{F}_s$-martingales on $s \in [0, t]$ (the series converge in L^2 uniformly in $s \le t$) with conditional quadratic variations

$$\langle N^{\xi}_{\cdot}(t, \varphi)\rangle_s = \kappa \sum_{y \in S} \int_0^s \xi_{r-}(y) \mathrm{E}\big[(P_{t-r}\varphi(Z+y) - P_{t-r}\varphi(y))^2\big]\, \mathrm{d}r$$
$$+ \sigma^2 \gamma \left(\sum_{x \in S} \int_0^s (P_{t-r}\varphi(x))^2\, \xi_{r-}(x)\eta_{r-}(x)\, \mathrm{d}r \right), \tag{5.3}$$

and

$$\langle N^{\eta}_{\cdot}(t, \varphi)\rangle_s = \kappa \sum_{y \in S} \int_0^s \eta_{r-}(y) \mathrm{E}\big[(P_{t-r}\varphi(Z+y) - P_{t-r}\varphi(y))^2\big]\, \mathrm{d}r$$
$$+ \sigma^2 \gamma \left(\sum_{x \in S} \int_0^s (P_{t-r}\varphi(x))^2\, \xi_{r-}(x)\eta_{r-}(x)\, \mathrm{d}r \right).$$

Here Z is the random variable distributed as a jump of the nearest-neighbor random walk.

Proof. The proof goes through application of Lemma 3.1 and Itô's formula to functions in the form of $f(s, \xi_s, \eta_s) = \langle \xi_s, P_{t-s}\varphi\rangle$ and $f(s, \xi_s, \eta_s) = \langle \eta_s, P_{t-s}\varphi\rangle$. The proof is pretty standard and we leave details to the enthusiastic reader.

In the end the orthogonality of the martingales $N^{\xi}_{\cdot}(t, \varphi)$ and $N^{\eta}_{\cdot}(t, \psi)$ follows from independence of driving Poisson point processes. □

Corollary 5.1. *Assume $S = \Lambda_n$. Let $(\xi_0, \eta_0) \in E_{\text{fin}} \times E_{\text{fin}}$. If $\varphi, \psi : S \to \mathbb{R}_+$, then*

$$\mathrm{E}\left(\langle \xi_t, \varphi\rangle\right) = \langle \xi_0, P_t\varphi\rangle\,, \qquad \mathrm{E}\left(\langle \eta_t, \psi\rangle\right) = \langle \eta_0, P_t\psi\rangle\,, \qquad \forall\, t \ge 0, \tag{5.4}$$

and

$$\mathrm{E}\left(\langle \xi_t, \varphi\rangle \langle \eta_t, \psi\rangle\right) = \langle \xi_0, P_t\varphi\rangle \langle \eta_0, P_t\psi\rangle\,, \qquad \forall\, t \ge 0. \tag{5.5}$$

Proof. Since Λ_n is finite, (5.4) follows immediately from Lemma 5.1.

As for (5.5), recalling again that Λ_n is finite, from Lemma 5.1 we get

$$\begin{aligned}
\mathrm{E}(\langle \xi_t, \varphi\rangle\langle \eta_t, \psi\rangle) &= \mathrm{E}\big(\big[\langle \xi_0, P_t\varphi\rangle + N^{\xi}_t(t, \varphi)\big]\big[\langle \eta_0, P_t\psi\rangle + N^{\eta}_t(t, \psi)\big]\big) \\
&= \langle \xi_0, P_t\varphi\rangle \langle \eta_0, P_t\psi\rangle + \langle \eta_0, P_t\psi\rangle\, \mathrm{E}\big(N^{\xi}_t(t, \varphi)\big) \\
&\quad + \langle \xi_0, P_t\varphi\rangle\, \mathrm{E}(N^{\eta}_t(t, \psi)) + \mathrm{E}\big(N^{\xi}_t(t, \varphi) N^{\eta}_t(t, \psi)\big) \\
&= \langle \xi_0, P_t\varphi\rangle \langle \eta_0, P_t\psi\rangle\,,
\end{aligned}$$

where the second and the third terms on the right-hand side are equal to zero since $N^{\xi}_t(t, \varphi)$ and $N^{\eta}_t(t, \psi)$ are martingales, and the last term vanishes because of the orthogonality of $N^{\xi}_t(t, \varphi)$ and $N^{\eta}_t(t, \psi)$. □

Let $B \subseteq S$ be an arbitrary finite (bounded) subset, and let $|B|$ denote a number of sites in B.

Now we are ready to compute the expected value and variance of a number of particles (each population separately) at a site $x \in S$ and at set B.

Corollary 5.2. *Assume $S = \Lambda_n$. Let $\xi_0(x) \equiv v$, $\eta_0(x) \equiv u\ \forall\, x \in S$, where $(v, u) \in \mathbb{N}_0^2$. Then,*

$$\mathrm{E}(\xi_t(x)) = v, \quad \mathrm{E}(\eta_t(x)) = u, \quad \text{and} \quad \mathrm{E}(\xi_t(x)\eta_t(y)) = vu \quad \forall\, x, y \in S,\ \forall\, t \geq 0.$$

Moreover, for any finite $B \subset S$,

$$\mathrm{E}(\xi_t(B)) = v\,|B|\,, \quad \text{and} \quad \mathrm{E}(\eta_t(B)) = u\,|B|\,, \quad \forall\, t > 0.$$

Proof. The result follows easily from Corollary 5.1. □

Before we treat the second moments of $\xi_t(x)$ and $\eta_t(x)$, let us prove a simple technical lemma. Recall that $g_t(\cdot, \cdot)$ is the Green function defined in (2.3) (for the nearest-neighbor random walk on $S = \Lambda_n$). Note that whenever the motion process is the nearest-neighbor random walk, we have $g_t(x, y) = g_t(x - y)$ (with certain abuse of notation).

Lemma 5.2. *Assume $S = \Lambda_n$, and the motion process is a nearest-neighbor random walk on S. For every $x \in S$:*

$$g_t(x) - \frac{1}{2d}\sum_{i=1}^{d}\left[g_t(x+e_i) + g_t(x-e_i)\right] = \frac{1}{\kappa}(\delta_{x,0} - p_t(x)), \quad \forall\, t \geq 0,$$

where $\delta_{x,y} = 1$ if and only if $x = y$ and $\delta_{x,y} = 0$ otherwise.

Proof. The proof follows a standard procedure, using the evolution equation for the transition densities of a continuous-time nearest-neighbor random walk. As we could not find an exact reference, we have included the derivation here for completeness.

The evolution of the transition probabilities of a continuous-time Markov chain is governed by the first-order differential equation

$$\frac{\partial}{\partial t}p_t(x) = Qp_t(x), \quad \forall\, t \geq 0,\ x \in S.$$

In our case $Q = (q_{xy})_{x,y\in S}$ where

$$q_{xx} = -\kappa, \quad q_{xy} = \frac{1}{2d}\kappa \quad \text{for } x \in S,\ y = x \pm e_i,\ i = 1, \ldots, d \quad \text{and} \quad q_{xy} = 0 \quad \text{otherwise.}$$

Therefore, we have

$$\begin{aligned}\frac{\partial}{\partial t}p_t(x) &= \sum_{y\in S} q_{xy}p_t(y) = q_{xx}(p_t(x) - \frac{1}{2d}\sum_{i=1}^{d}(p_t(x+e_1) + p_t(x-e_1)))\\ &= -\kappa\left(p_t(x) - \frac{1}{2d}\sum_{i=1}^{d}\left[p_t(x+e_i) + p_t(x-e_i)\right]\right).\end{aligned}$$

Now, integrate both sides over the interval $[0, t]$, which gives

$$\begin{aligned}p_t(x) - p_0(x) &= -\kappa\int_0^t\left(p_s(x) - \frac{1}{2d}\sum_{i=1}^{d}\left[p_s(x+e_i) + p_s(x-e_i)\right]\right)\mathrm{d}s\\ &= -\kappa\left(g_t(x) - \frac{1}{2d}\sum_{i=1}^{d}\left[g_t(x+e_i) + g_t(x-e_i)\right]\right).\end{aligned}$$

The result follows immediately once we recall that $p_0(x) = \delta_{x,0}$. □

Now we are ready to handle the second moments of $\xi_t(x)$ and $\eta_t(x)$.

Lemma 5.3. *Assume* $S = \Lambda_n$. *Let* $\xi_0(x) \equiv v, \ \ \eta_0(x) \equiv u$ *for all* $x \in S$, *where* $(v, u) \in \mathbb{N}_0^2$. *Then, for all* $t \geq 0$,

$$\mathrm{E}\big(\xi_t(x)^2\big) = v^2 + \frac{1}{2}\sigma^2 \gamma uv g_{2t}(0) + v(1 - p_{2t}(0)), \tag{5.6}$$

and

$$\mathrm{E}(\eta_t(x)^2) = u^2 + \frac{1}{2}\sigma^2 \gamma uv g_{2t}(0) + u(1 - p_{2t}(0)). \tag{5.7}$$

Proof. We prove only (5.6), since the proof of (5.7) is the same. Again we use the representation of the process from Lemma 5.1, with $\varphi(\cdot) = \delta_x(\cdot)$ and use notation $\phi_r(\cdot) = p_{t-r}\delta_x(\cdot) = p_{t-r}(\cdot - x)$.

For $0 \leq s \leq t$ denote

$$\begin{aligned}
N_s^\xi(y) &= N_s^\xi(t, y) = N_s^\xi(t, \delta_y). \\
\mathrm{E}\big(\xi_t(x)^2\big) &= (P_t\xi_0(x))^2 + \mathrm{E}\big(\big(N_t^\xi(x)\big)^2\big) \\
&= v^2 + \mathrm{E}\left(\langle N_\cdot^\xi(x)\rangle_t\right) \\
&= v^2 + v\kappa \sum_{y\in S} \int_0^t \sum_{z \in S} (\phi_r(z) - \phi_r(y))^2 p_{y,z}\,\mathrm{d}r \\
&\quad + \sigma^2 \gamma uv \sum_{y \in S} \int_0^t \phi_r(y)^2\,\mathrm{d}r \\
&=: v^2 + v\kappa J_1(t) + \sigma^2\gamma uv J_2(t), \qquad t \geq 0
\end{aligned}$$

where in the third equality we used again Lemma 5.1, the Fubini theorem, and Corollary 5.1 which implies $\mathrm{E}(\xi_{r-}(y)\eta_{r-}(y)) = P_t\xi_0(y)P_t\eta_0(y) = vu$, $\mathrm{E}(\xi_{r-}(y)) = P_t\xi_0(y) = v$. Now we will compute each term separately.

First, let us evaluate $J_2(t)$. Recall that $\phi_r(y) = p_{t-r}(y - x)$. Then we have

$$J_2(t) = \int_0^t \sum_{y \in S} p_{t-r}(y-x)^2\,\mathrm{d}r = \int_0^t p_{2(t-r)}(0)\,\mathrm{d}r = \frac{1}{2}\int_0^{2t} p_\tau(0)\,\mathrm{d}\tau = 0.5 g_{2t}(0), \tag{5.8}$$

where $p_s(x, x) = p_s(0, 0) = p_s(0)$, for all $x \in S$.

Now we will handle $J_1(t)$:

$$\begin{aligned}
J_1(t) &= \sum_{y \in S} \int_0^t \sum_{z\in S} (\phi_r(z) - \phi_r(y))^2 p_{y,z}\,\mathrm{d}r \qquad (5.9)\\
&= \sum_{y\in S}\sum_{z\in S}\int_0^t p_{t-r}(z-x)^2 p_{y,z}\,\mathrm{d}r + \sum_{y\in S}\sum_{z\in S}\int_0^t p_{t-r}(y-x)^2 p_{y,z}\,\mathrm{d}\mathrm{d}r \\
&\quad - 2\sum_{y\in S}\sum_{z\in S}\int_0^t p_{t-r}(z-x)p_{t-r}(y-x)p_{y,z}\,\mathrm{d}r.
\end{aligned}$$

We will treat each of the three terms above separately. For the first term we have

$$\sum_{y\in S}\sum_{z\in S}\int_0^t p_{t-r}(z,x)^2 p_{y,z}\,\mathrm{d}r=\sum_{z\in S}\int_0^t p_{t-r}(z,x)^2\,\mathrm{d}r=0.5g_{2t}(0) \tag{5.10}$$

where the last equality follows as in (5.8).

Similarly we get

$$\sum_{y\in S}\sum_{z\in S}\int_0^t p_{t-r}(y,x)^2 p_{y,z}\,\mathrm{d}\mathrm{d}r=0.5g_{2t}(0). \tag{5.11}$$

Finally. it is easy to obtain

$$\begin{aligned}\sum_{y\in S}\sum_{z\in S}\int_0^t & p_{t-r}(z,x)p_{t-r}(y,x)p_{y,z}\,\mathrm{d}r\\ &=0.5\frac{1}{2d}\sum_{i=1}^d\big[g_{2t}(e_i)+g_{2t}(-e_i)\big]=\frac{1}{2d}\sum_{i=1}^d g_{2t}(e_i).\end{aligned} \tag{5.12}$$

By putting (5.10), (5.11), (5.12), and (5.8) together we have

$$\mathrm{E}\big(\xi_t(x)^2\big)=v^2+\sigma^2\gamma uv\frac{1}{2}g_{2s}(0)+v\kappa\left(g_{2t}(0)-\frac{1}{2d}\sum_{i=1}^d g_{2t}(e_i)\right),\qquad t\ge 0,\ x\in S. \tag{5.13}$$

Now use (5.13) and Lemma 5.2 with $x=y$ to get

$$\mathrm{E}\big(\xi_t(x)^2\big)=v^2+\frac{1}{2}\sigma^2\gamma uvg_{2t}(0)+v(1-p_{2t}(0)),\qquad \forall\, t\ge 0,\ x\in S.$$ □

We also need to evaluate $\mathrm{E}(\xi_t(x)\xi_t(y))$ for $x\neq y$. To this end we will prove the following lemma.

Lemma 5.4. *Assume* $S=\Lambda_n$. *Let* $\xi_0(x)\equiv v$, $\eta_0(x)\equiv u\ \forall\ x\in S$, *where* $(v,u)\in\mathbb{N}_0^2$. *Let* $x\neq y$, *then*

$$\mathrm{E}(\xi_t(x)\xi_t(y))=v^2-vp_{2t}(x-y)+\frac{1}{2}\sigma^2\gamma uvg_{2t}(x-y),\qquad \forall\, t\ge 0,$$

and

$$\mathrm{E}(\eta_t(x)\eta_t(y))=u^2-up_{2t}(x-y)+\frac{1}{2}\sigma^2\gamma uvg_{2t}(x-y),\qquad \forall\, t\ge 0.$$

Proof. The proof goes along similar lines as the proof of Lemma 5.3 and thus is omitted. □

6. Proof of Theorem 2.3

Let $\big(\xi_t^n,\eta_t^n\big)$ be a pair of processes solving (2.4) with site space $S=\Lambda_n$, and $N_{x,y}^{\mathrm{RW}_\xi}$, $N_{x,y}^{\mathrm{RW}_\eta}$ being Poisson point processes with intensity measure $q^n(x,y)\,\mathrm{d}s\otimes\mathrm{d}u$, q^n is defined by (1.2). Here $\{p_{x,y}^n\}_{x,y\in\Lambda_n}$ are the transition jump probabilities of the underlying random walk, and $\big\{P_t^n\big\}_{t\ge 0}$ is the associated semigroup. In what follows, we assume $d\ge 3$.

Fix $(\theta_1, \theta_2) \in \mathbb{N}_0^2$. Assume the following initial conditions for (ξ_t^n, η_t^n) :

$$\xi_0^n(x) = \theta_1, \quad \eta_0^n(x) = \theta_2 \quad \forall\, x \in \Lambda_n.$$

Set

$$\boldsymbol{\xi}_t^n = \sum_{j\in\Lambda_n} \xi_j^n(t), \quad \boldsymbol{\eta}_t^n = \sum_{j\in\Lambda_n} \eta_j^n(t). \tag{6.1}$$

We define the following time change:

$$\beta_n(t) = |\Lambda_n|\, t, \quad t \geq 0.$$

Theorem 2.3 identifies the limiting distribution of

$$\frac{1}{|\Lambda_n|}\big(\boldsymbol{\xi}^n_{\beta_n(t)}, \boldsymbol{\eta}^n_{\beta_n(t)}\big),$$

as $n \to \infty$, for $t \in [0, 1]$.

In Section 1 we defined a system of Dawson–Perkins processes $(U_t^n, V_t^n)_{t\geq 0}$ on Λ_n, that solves (1.3). Recall that

$$\mathbf{U}_t^n = \sum_{i\in\Lambda_n} u_t^n(i), \quad \mathbf{V}_t^n = \sum_{i\in\Lambda_n} v_t^n(i).$$

The limiting behavior of $(\mathbf{U}_t^n, \mathbf{V}_t^n)_{t\geq 0}$ was studied in [6], we stated the result in Theorem 1.1.

Theorem 2.3 claims that the limiting behavior of $\frac{1}{|\Lambda_n|}\big(\boldsymbol{\xi}^n_{\beta_n(t)}, \boldsymbol{\eta}^n_{\beta_n(t)}\big)$ is similar to $\frac{1}{|\Lambda_n|}\big(\mathbf{U}^n_{\beta_n(t)}, \mathbf{V}^n_{\beta_n(t)}\big)$ for $t \in [0, 1]$. As we have mentioned above, in contrast to Dawson–Perkins processes solving equation (1.3), the useful self-duality property does not hold for our branching particle model. However, we use the so-called approximating duality technique that allows us to prove Theorem 2.3.

In what follows, we will use a periodic sum on Λ_n: for $x, y \in \Lambda_n$ we have $x + y = (x + y)\Lambda(\mathrm{mod})\, n \in \Lambda_n$.

The next proposition is crucial for the proof of Theorem 2.3.

Proposition 6.1. *Let $(X_t, Y_t)_{t\geq 0}$ be the solution to (2.9). Then for all $a, b \geq 0$,*

$$\lim_{n\to\infty} \mathrm{E}\Big(\mathrm{e}^{-\frac{1}{|\Lambda_n|}(\boldsymbol{\xi}^n_{\beta_n(t)}+\boldsymbol{\eta}^n_{\beta_n(t)})(a+b)-\mathrm{i}\frac{1}{|\Lambda_n|}(\boldsymbol{\xi}^n_{\beta_n(t)}-\boldsymbol{\eta}^n_{\beta_n(t)})(a-b)}\Big) = \mathrm{E}\Big(\mathrm{e}^{-(X_t+Y_t)(a+b)-\mathrm{i}(X_t-Y_t)(a-b)}\Big),$$

for $t \in [0, 1]$.

Proof of Theorem 2.3. By easy adaptation of Lemma 2.5 of [33] one gets that the mixed Laplace–Fourier transform

$$\mathrm{E}\Big(\mathrm{e}^{-(X+Y)(a+b)-\mathrm{i}(X-Y)(a-b)}\Big), \quad a, b \geq 0,$$

determines the distribution of non-negative two-dimensional random variables (X, Y). Therefore, Theorem 2.3 follows easily from Proposition 6.1 and properties of weak convergence.

The rest of the section is organized as follows. Section 6.1 is devoted to the proof of Proposition 6.1, and the proof of one of the technical propositions is deferred to Section 6.2. □

6.1. Proof of Proposition 6.1

In what follows, fix $T \in (0, 1]$. Let $(\xi_t^n, \eta_t^n)_{t\geq 0}$ be a mutually catalytic branching random walk from Theorem 2.3 (we will refer to it as the 'discrete process'). In the proof of the proposition we will use the duality technique introduced in [33]. To this end, we need the following Dawson–Perkins processes.

- Let $(u_t^n, v_t^n)_{t\geq 0}$ be a solution to (1.3), with Q^n being the Q-matrix of the nearest-neighbor random walk on Λ_n, and with some initial conditions (u_0, v_0).
- For arbitrary $a, b \geq 0$, the sequence $(\tilde{u}_t^n, \tilde{v}_t^n)_{t\geq 0}$ solving (1.3) with initial conditions

$$\tilde{u}_0^n(x) = \frac{a}{|\Lambda_n|}, \quad \tilde{v}_0^n(x) = \frac{b}{|\Lambda_n|} \quad \text{for every } x \in \Lambda_n. \tag{6.2}$$

In what follows, we assume that $(u_t^n, v_t^n)_{t\geq 0}$, $(\tilde{u}_t^n, \tilde{v}_t^n)_{t\geq 0}$ and $(\xi_t^n, \eta_t^n)_{t\geq 0}$ are independent. Now let us describe the state spaces for the processes involved in this section.

Similarly to E_{fin} define $E_{\text{fin}}^n = \{f : \Lambda_n \longrightarrow \mathbb{N}_0\}$, and $E_{\text{fin,con}}^n = E_{\text{fin}}^n \times E_{\text{fin}}^n$. Clearly, since Λ_n is compact, the L^1 norm of functions in E_{fin}^n is finite. In addition, define $\widetilde{E}_{\text{fin}}^n = \{f : \Lambda_n \longrightarrow \mathbb{R}_+\}$, and $\widetilde{E}_{\text{fin,con}}^n = \widetilde{E}_{\text{fin}}^n \times \widetilde{E}_{\text{fin}}^n$.

First, by Theorem 2.1, the process (ξ_t^n, η_t^n) that solves (2.4) with initial conditions $(\xi_0^n, \eta_0^n) = \bar{\boldsymbol{\theta}}$ is an $E_{\text{fin}}^n \times E_{\text{fin}}^n$-valued process. By our definition (6.2), $(\tilde{u}_0^n, \tilde{v}_0^n) \in \widetilde{E}_{\text{fin,con}}^n$. Moreover, by simple adaptation of the proof of Theorem 2.2(d) in [16] to our state space Λ_n, we get

$$\left(\tilde{u}_t^n, \tilde{v}_t^n\right) \in \widetilde{E}_{\text{fin,con}}, \quad \forall\, t \geq 0.$$

For $(\varphi, \psi, \tilde{\varphi}, \tilde{\psi}) \in \mathbb{R}_+^{\Lambda_n} \times \mathbb{R}_+^{\Lambda_n} \times \mathbb{R}_+^{\Lambda_n} \times \mathbb{R}_+^{\Lambda_n}$ define

$$H(\varphi, \psi, \tilde{\varphi}, \tilde{\psi}) = \mathrm{e}^{-\langle \varphi+\psi, \tilde{\varphi}+\tilde{\psi}\rangle - \mathrm{i}\langle \varphi-\psi, \tilde{\varphi}-\tilde{\psi}\rangle},$$

and

$$\begin{aligned} F_{t,s}^n &= \mathrm{E}\left[H\left(\xi_t^n, \eta_t^n, \tilde{u}_s^n, \tilde{v}_s^n\right)\right] \\ &= \mathrm{E}\left[\mathrm{e}^{-\langle \xi_t^n+\eta_t^n, \tilde{u}_s^n+\tilde{v}_s^n\rangle - \mathrm{i}\langle \xi_t^n-\eta_t^n, \tilde{u}_s^n-\tilde{v}_s^n\rangle}\right], \end{aligned}$$

for $0 \leq s, t \leq \beta_n(T)$.

Let us recall the self-duality lemma from [6, Lemma 4.1].

Lemma 6.1. *Let* $(u_0, v_0), (\tilde{u}_0, \tilde{v}_0) \in \widetilde{E}_{\text{fin,con}}^n$, *where* $(u_t, v_t)_{t\geq 0}, (\tilde{u}_t, \tilde{v}_t)_{t\geq 0}$ *are independent solutions of* (1.3). *Then*

$$\mathrm{E}\left(H(u_t, v_t, \tilde{u}_0, \tilde{v}_0)\right) = \mathrm{E}\left(H(u_0, v_0, \tilde{u}_t, \tilde{v}_t)\right).$$

Remark 6.1. In [6] the above lemma is proved for more general state spaces and initial conditions. The conditions in Lemma 4.1 in [6] hold trivially in our case.

Then we have the following proposition.

Proposition 6.2. *For any* $(\theta_1, \theta_2) \in \mathbb{N}_0^2$, $a, b \geq 0$,

$$\lim_{n\to\infty} \mathrm{E}\left[F_{\beta_n(T),0}^n\right] = \lim_{n\to\infty} \mathrm{E}\left[F_{0,\beta_n(T)}^n\right]. \tag{6.3}$$

Proof. The proof is postponed until the end of this section. It is proved via a series of other results. □

Given Proposition 6.2, it is easy to complete.

Proof of Proposition 6.1. Fix arbitrary $\theta_1, \theta_2 \geq 0$ and $a, b \geq 0$. For any $n \geq 1$, let $(u^n_t, v^n_t)_{t\geq 0}$ be the solution to (1.3) with Q^n being a Q-matrix of the nearest-neighbor random walk on Λ_n, and initial conditions $(u^n_0, v^n_0) = (\xi^n_0, \eta^n_0) = \bar{\boldsymbol{\theta}}$. Recall that

$$U^n_t = \sum_{x\in\Lambda_n} u^n_t(x), \qquad V^n_t = \sum_{x\in\Lambda_n} v^n_t(x).$$

Note that

$$\begin{aligned}\lim_{n\to\infty} \mathrm{E}\big[F^n_{0,\beta_n(T)}\big] &= \lim_{n\to\infty} \mathrm{E}\Big(\mathrm{e}^{-\left\langle \xi^n_0+\eta^n_0, \tilde{u}^n_{\beta_n(T)}+\tilde{v}^n_{\beta_n(T)}\right\rangle - \mathrm{i}\langle \xi^n_0-\eta^n_0, \tilde{u}^n_{\beta_n(T)}-\tilde{v}^n_{\beta_n(T)}\rangle}\Big)\\ &= \lim_{n\to\infty} \mathrm{E}\Big(\mathrm{e}^{-\left(\mathbf{U}^n_{\beta_n(T)}+\mathbf{V}^n_{\beta_n(T)}\right)\frac{1}{|\Lambda_n|}(a+b) - \mathrm{i}\left(\mathbf{U}^n_{\beta_n(T)}-\mathbf{V}^n_{\beta_n(T)}\right)\frac{1}{|\Lambda_n|}(a-b)}\Big)\\ &= \mathrm{E}\big(e^{-(X_T+Y_T)(a+b)-i(X_T-Y_T)(a-b)}\big),\end{aligned} \tag{6.4}$$

where the second equality follows by a self-duality relation in Lemma 6.1, and the third equality follows by Theorem 1.1. This means that

$$\lim_{n\to\infty} \mathrm{E}\left(\mathrm{e}^{-(\boldsymbol{\xi}^n_{\beta(T)}+\boldsymbol{\eta}^n_{\beta_n(T)})\frac{1}{|\Lambda_n|}(a+b) - \mathrm{i}(\boldsymbol{\xi}^n_{\beta_n(T)}-\boldsymbol{\eta}^n_{\beta_n(T)})\frac{1}{|\Lambda_n|}(a-b)}\right) \tag{6.5}$$

$$\begin{aligned}&= \lim_{n\to\infty} \mathrm{E}\big[F^n_{\beta_n(T),0}\big] = \lim_{n\to\infty} \mathrm{E}\big[F^n_{0,\beta_n(T)}\big]\\ &= \mathrm{E}\big(\mathrm{e}^{-(X_T+Y_T)(a+b)-\mathrm{i}(X_T-Y_T)(a-b)}\big),\end{aligned} \tag{6.6}$$

where the second equality follows by Proposition 6.2, and the last equality follows by (6.4). This finishes the proof of Proposition 6.1. □

To prove Proposition 6.2 we will need other results. First we need [21, Lemma 4.10].

Lemma 6.2. (Lemma 4.10 of [21]). *Suppose a function f(s,t) on* $[0, \infty) \times [0, \infty)$ *is absolutely continuous in s for each fixed t and absolutely continuous in t for each fixed s. Set* $(f_1, f_2) \equiv \nabla f$*, and assume that*

$$\int_0^T \int_0^T |f_i(s, t)| \, \mathrm{d}s \, \mathrm{d}t < \infty, \quad i = 1, 2, \ \forall\, T > 0. \tag{6.7}$$

Then for almost every $t \geq 0$,

$$f(t, 0) - f(0, t) = \int_0^t (f_1(s, t-s) - f_2(s, t-s)) \, \mathrm{d}s. \tag{6.8}$$

We apply this lemma for the function $F^n_{r,s} = \mathrm{E}\left[H\left(\xi^n_r, \eta^n_r, \tilde{u}^n_s, \tilde{v}^n_s\right)\right]$. Then we show that for $f(r, s) = F^n_{r,s}$ and $t = \beta_n(T)$, the right-hand side of (6.8) tends to 0, as $n \to \infty$.

In order to check the conditions in Lemma 6.2 we will need several lemmas. In the next two lemmas we will derive martingale problems for processes $(\xi^n_\cdot, \eta^n_\cdot)$ and $(\tilde{u}^n_\cdot, \tilde{v}^n_\cdot)$. Recall that

$\{p^n_{x,y}\}_{x,y\in\Lambda_n}$ are the transition jump probabilities of the underlying nearest-neighbor random walk on Λ_n.

Lemma 6.3. *For any $(\varphi, \psi) \in \widetilde{E}^n_{\text{fin,con}}$ define*

$$\begin{aligned} g\left(\xi^n_s, \eta^n_s, \varphi, \psi\right) = H\left(\xi^n_s, \eta^n_s, \varphi, \psi\right) \Bigg\{ &\kappa \sum_{x,y\in\Lambda_n} \xi^n_s(x) \\ &\times p^n_{xy}\left[\mathrm{e}^{-\varphi(y)-\psi(y)+\varphi(x)+\psi(x)-\mathrm{i}(\varphi(y)-\psi(y)-\varphi(x)+\psi(x))} - 1\right] \\ &+ \kappa \sum_{x,y\in\Lambda_n} \eta^n_s(x) p^n_{xy}\left[\mathrm{e}^{-\varphi(y)-\psi(y)+\varphi(x)+\psi(x)+\mathrm{i}(\varphi(y)-\psi(y)-\varphi(x)+\psi(x))} - 1\right] \\ &+ \gamma \sum_{x\in\Lambda_n} \xi^n_s(x)\eta^n_s(x) \sum_{k\geq 0} \nu_k\left[\mathrm{e}^{-(k-1)(\varphi(x)+\psi(x)+\mathrm{i}(\varphi(x)-\psi(x)))} - 1\right] \\ &+ \gamma \sum_{x\in\Lambda_n} \xi^n_s(x)\eta^n_s(x) \sum_{k\geq 0} \nu_k\left[\mathrm{e}^{-(k-1)(\varphi(x)+\psi(x)-\mathrm{i}(\varphi(x)-\psi(x)))} - 1\right] \Bigg\}, \quad \forall\, s \geq 0. \end{aligned} \tag{6.9}$$

Then

$$H\left(\xi^n_t, \eta^n_t, \varphi, \psi\right) - \int_0^t g\left(\xi^n_s, \eta^n_s, \varphi, \psi\right) \mathrm{d}s, \quad \forall\, t \geq 0.$$

is an $\{\mathcal{F}^{\xi,\eta}_t\}_{t\geq 0}$-martingale.

Proof. The result is immediate by Lemma 3.1(d). □

A similar result holds for the Dawson–Perkins process.

Lemma 6.4. *For any $(\varphi, \psi) \in E^n_{\text{fin,con}}$, define*

$$\begin{aligned} h\left(\varphi, \psi, \tilde{u}^n_s, \tilde{v}^n_s\right) = H\left(\varphi, \psi, \tilde{u}^n_s, \tilde{v}^n_s\right) \Bigg\{ &- \sum_{x\in\Lambda_n} \tilde{u}^n_s Q^n(x)\left(\varphi(x) + \psi(x) + i\left(\varphi(x) - \psi(x)\right)\right) \\ &- \sum_{x\in\Lambda_n} \tilde{v}^n_s Q^n(x)\left(\varphi(x) + \psi(x) - i\left(\varphi(x) - \psi(x)\right)\right) \\ &+ 4\tilde{\gamma} \sum_{x\in\Lambda_n} \tilde{u}^n_s(x)\tilde{v}^n_s(x)\varphi(x)\psi(x) \Bigg\}, \quad \forall\, s \geq 0. \end{aligned} \tag{6.10}$$

Then

$$H\left(\varphi, \psi, \tilde{u}^n_t, \tilde{v}^n_t\right) - \int_0^t h\left(\varphi, \psi, \tilde{u}^n_s, \tilde{v}^n_s\right) \mathrm{d}s, \quad t \geq 0,$$

is an $\{\mathcal{F}^{\tilde{u}^n,\tilde{v}^n}_t\}_{t\geq 0}$-martingale.

Proof. The result is immediate by [16, Theorem 2.2(c)(iv)], Itô's lemma [24, Theorem II.5.1]), and simple algebra.

Lemma 6.5. *For any $t > 0$,*

$$\sup_{\substack{0 \le s \le t \\ 0 \le r \le t}} \mathrm{E}\left|h(\xi_r^n, \eta_r^n, \tilde{u}_s^n, \tilde{v}_s^n)\right| < \infty \tag{6.11}$$

and

$$\sup_{\substack{0 \le s \le t \\ 0 \le r \le t}} \mathrm{E}\left|g(\xi_r^n, \eta_r^n, \tilde{u}_s^n, \tilde{v}_s^n)\right| < \infty. \tag{6.12}$$

Proof. Equation (6.11) is verified in the proof of Theorem 2.4(b) in [16].

Now let us check (6.12). First, by simple algebra it is trivial to see that for any $z \in \mathbb{R}_+$ and $y \in \mathbb{R}$,

$$\left|\mathrm{e}^{-z+\mathrm{i}y} - 1\right| \le (|z| + |y|) . \tag{6.13}$$

Hence,

$$\begin{aligned}
&\sup_{\substack{0 \le s \le t \\ 0 \le r \le t}} \mathrm{E}\left|g\left(\xi_r^n, \eta_r^n, \tilde{u}_s^n, \tilde{v}_s^n\right)\right| \\
&\qquad \le \sup_{0 \le s, r \le t} C\mathrm{E}\Bigg\{\kappa \sum_{x,y \in \Lambda_n} \xi_r^n(x) p_{xy}^n \left[\tilde{u}_s^n(y) + \tilde{v}_s^n(y) + \tilde{u}_s^n(x) + \tilde{v}_s^n(x)\right] \\
&\qquad + \kappa \sum_{x,y \in \Lambda_n} \eta_r^n(x) p_{xy}^n \left[\tilde{u}_s^n(y) + \tilde{v}_s^n(y) + \tilde{u}_s^n(x) + \tilde{v}_s^n(x)\right] \\
&\qquad + \gamma \sum_{x \in \Lambda_n} \xi_r^n(x) \eta_r^n(x) \sum_{k \ge 0} \nu_k \, |k-1| \left[\tilde{u}_s^n(x) + \tilde{v}_s^n(x)\right] \\
&\qquad + \gamma \sum_{x \in \Lambda_n} \xi_r^n(x) \eta_r^n(x) \sum_{k \ge 0} \nu_k \, |k-1| \left[\tilde{u}_s^n(x) + \tilde{v}_s^n(x)\right] \Bigg\},
\end{aligned}$$

where $C > 0$ is a constant and the last inequality follows from (6.13). Recall that, by Corollary 5.2, $\mathrm{E}[\xi_s^n(x)] = \theta_1$, $\mathrm{E}[\eta_s^n(x)] = \theta_2$ and, $\mathrm{E}[\xi_s^n(x)\eta_s^n(x)] = \theta_1\theta_2$. By [16, Theorem 2.2b(iii)],

$$\mathrm{E}\left[\langle \tilde{u}_s^n, 1 \rangle\right] = a < \infty, \quad \mathrm{E}\left[\langle \tilde{v}_s^n, 1 \rangle\right] = b < \infty, \quad \forall\, s \ge 0,$$

since initial conditions have a finite mass. Also note that $\sum_{k \ge 0} |k-1| \, \nu_k < \infty$. Then (6.12) holds. □

Now we are ready to prove the following lemma.

Lemma 6.6. *For any $n \ge 1$, and every $t > 0$,*

$$\begin{aligned}
&\mathrm{E}\left[H\left(\xi_t^n, \eta_t^n, \tilde{u}_0^n, \tilde{v}_0^n\right)\right] - \mathrm{E}\left[H\left(\xi_0^n, \eta_0^n, \tilde{u}_t^n, \tilde{v}_t^n\right)\right] \\
&\quad = \mathrm{E}\left[\int_0^t \left\{g\left(\xi_s^n, \eta_s^n, \tilde{u}_{t-s}^n, \tilde{v}_{t-s}^n\right) - h\left(\xi_s^n, \eta_s^n, \tilde{u}_{t-s}^n, \tilde{v}_{t-s}^n\right)\right\} \mathrm{d}s\right].
\end{aligned} \tag{6.14}$$

Proof. By Lemmas 6.3, 6.4, and 6.5 we can apply Lemma 6.2 to the function

$$F^n_{r,s} = \mathrm{E}\left[H\left(\xi^n_r, \eta^n_r, \tilde{u}^n_s, \tilde{v}^n_s\right)\right],$$

and immediately see that (6.14) holds for almost every $t > 0$. However, again by Lemmas 6.3, 6.4, and 6.5 one can see that both left-hand and right-hand sides of (6.14) are continuous in t. Hence, (6.14) holds for all $t > 0$. □

Define

$$e(T, n) = \mathrm{E}\Bigg[\int_0^{\beta_n(T)} \Big\{g\Big(\xi^n_s, \eta^n_s, \tilde{u}^n_{\beta_n(T)-s}, \tilde{v}^n_{\beta_n(T)-s}\Big) - h\Big(\xi^n_s, \eta^n_s, \tilde{u}^n_{\beta_n(T)-s}, \tilde{v}^n_{\beta_n(T)-s}\Big)\Big\}\,\mathrm{d}s\Bigg]. \tag{6.15}$$

To finish the proof of Proposition 6.2 we need the following proposition.

Proposition 6.3. *We have $e(T, n) \to 0$ as $n \to \infty$.*

The next subsection is devoted to the proof of the above proposition. Now we are ready to complete.

Proof of Proposition 6.2. The proof is immediate by Lemma 6.6 and Proposition 6.3. □

6.2. Proof of Proposition 6.3

Fix $t > 0$. For simplicity, denote $f_s = H(\xi^n_s, \eta^n_s, \varphi, \psi)$. Apply the Taylor series expansion on the exponents inside the sums on the right-hand side of (6.9) to get

$$\begin{aligned}
&g(\xi^n_s, \eta^n_s, \varphi, \psi)\\
&\quad = f_s\Bigg\{\kappa \sum_{x,y\in\Lambda_n} \xi^n_s(x)p^n_{xy}\\
&\qquad \times \big[-\varphi(y) - \psi(y) + \varphi(x) + \psi(x) - \mathrm{i}\,(\varphi(y) - \psi(y) - \varphi(x) + \psi(x))\\
&\qquad + \frac{1}{2}\,(-\varphi(y) - \psi(y) + \varphi(x) + \psi(x) - \mathrm{i}\,(\varphi(y) - \psi(y) - \varphi(x) + \psi(x)))^2\\
&\qquad + G^{1,1}(\varphi, \psi, x, y)\big]\Bigg\}\\
&\qquad + f_s\Bigg\{\kappa \sum_{x,y\in\Lambda_n} \eta^n_s(x)p^n_{xy}\\
&\qquad \times \big[-\varphi(y) - \psi(y) + \varphi(x) + \psi(x) + \mathrm{i}\,(\varphi(y) - \psi(y) - \varphi(x) + \psi(x))\\
&\qquad + \frac{1}{2}\,(-\varphi(y) - \psi(y) + \varphi(x) + \psi(x) + \mathrm{i}\,(\varphi(y) - \psi(y) - \varphi(x) + \psi(x)))^2\\
&\qquad + G^{1,2}(\varphi, \psi, x, y)\big]\\
&\qquad + \gamma \sum_{x\in\Lambda_n} \xi^n_s(x)\eta^n_s(x)\left[\frac{1}{2}\sigma^2\,(\varphi(x) + \psi(x) + \mathrm{i}\,(\varphi(x) - \psi(x)))^2 + G^{2,1}(\varphi, \psi, x)\right]\\
&\qquad + \gamma \sum_{x\in\Lambda_n} \xi^n_s(x)\eta^n_s(x)\left[\frac{1}{2}\sigma^2\,(\varphi(x) + \psi(x) - \mathrm{i}\,(\varphi(x) - \psi(x)))^2 + G^{2,2}(\varphi, \psi, x)\right]\Bigg\},\\
&\qquad \forall s \geq 0.
\end{aligned}$$

where we used our assumption on the branching mechanism:

$$\sum_{k\geq 0} \nu_k (k-1) = 0 \quad \text{and} \quad \sum_{k\geq 0} \nu_k (k-1)^2 = \sigma^2.$$

For the error terms in the Taylor expansion we have the following bounds

$$\begin{aligned} |G^{1,j}(\varphi, \psi, x, y)| &\leq \mathrm{e}^{\varphi(x)+\psi(x)} \left| -\varphi(y) - \psi(y) + \varphi(x) + \psi(x) - \mathrm{i}\,(\varphi(y) - \psi(y) - \varphi(x) + \psi(x)) \right|^3 \\ &\leq C_{(6.16)} \mathrm{e}^{\varphi(x)+\psi(x)} \big(\varphi(y)^3 + \psi(y)^3 + \varphi(x)^3 + \psi(x)^3\big), \quad j = 1, 2, \end{aligned} \tag{6.16}$$

$$\begin{aligned} |G^{2,1}(\varphi, \psi, x)| + |G^{2,2}(\varphi, \psi, x)| &\leq \mathrm{e}^{\varphi(x)+\psi(x)} \sum_{k\geq 0} \nu_k |k-1|^3 \big(|\varphi(x) + \psi(x) - i\,(\varphi(x) - \psi(x))|^3 \\ &\qquad + |\varphi(x) + \psi(x) + i\,(\varphi(x) - \psi(x))|^3 \big) \\ &\leq C_{(6.17)} \mathrm{e}^{\varphi(x)+\psi(x)} \big(\varphi(x)^3 + \psi(x)^3\big), \end{aligned} \tag{6.17}$$

where the positive constants $C_{(6.16)}$, $C_{(6.17)}$ are independent of φ, ψ, x, y and in (6.17) we used the assumption on $\sum_{k\geq 0} \nu_k k^3 < \infty$ on the branching mechanism.

We use simple algebra to obtain

$$\begin{aligned} g(\xi_s^n, \eta_s^n, \varphi, \psi) = f_s \Big\{ &\kappa \sum_{x,y\in\Lambda_n} \xi_s^n(x) p_{xy}^n \\ &\times \big[-\varphi(y) - \psi(y) + \varphi(x) + \psi(x) - \mathrm{i}\,(\varphi(y) - \psi(y) - \varphi(x) + \psi(x)) \\ &+ 2\,(\varphi(x) - \varphi(y))\,(\psi(x) - \psi(y)) + i\big((\varphi(x) - \varphi(y))^2 - (\psi(x) - \psi(y))^2\big) \\ &+ G^{1,1}(\varphi, \psi, x, y) \big] \Big\} \\ &+ f_s \Big\{ \kappa \sum_{x,y\in\Lambda_n} \eta_s^n(x) p_{xy}^n \\ &\times \big[-\varphi(y) - \psi(y) + \varphi(x) + \psi(x) + i\,(\varphi(y) - \psi(y) - \varphi(x) + \psi(x)) \\ &+ 2\,(\varphi(x) - \varphi(y))\,(\psi(x) - \psi(y)) - i\big((\varphi(x) - \varphi(y))^2 - (\psi(x) - \psi(y))^2\big) \\ &+ G^{1,2}(\varphi, \psi, x, y) \big] \\ &+ \gamma \sum_{x\in\Lambda_n} \xi_s^n(x) \eta_s^n(x) \big[4\sigma^2 \varphi(x)\psi(x) + G^{2,1}(\varphi, \psi, x) \\ &+ G^{2,2}(\varphi, \psi, x) \big] \Big\}, \quad \forall\, s \geq 0. \end{aligned}$$

Let us define

$$\begin{aligned} \tilde{f}_{T,s}^n &= H\big(\xi_s^n, \eta_s^n, \tilde{u}_{\beta_n(T)-s}^n, \tilde{v}_{\beta_n(T)-s}^n\big) \\ &= \mathrm{e}^{-\left\langle \xi_s^n + \eta_s^n, \tilde{u}_{\beta_n(T)-s}^n + \tilde{v}_{\beta_n(T)-s}^n \right\rangle - \mathrm{i} \left\langle \xi_s^n - \eta_s^n, \tilde{u}_{\beta_n(T)-s}^n - \tilde{v}_{\beta_n(T)-s}^n \right\rangle}, \quad 0 \leq s \leq T. \end{aligned} \tag{6.18}$$

Now by using the above and Lemmas 6.4 and 6.3 we get (recall that $\tilde{\gamma} = \gamma\sigma^2$ and $e(T, n)$ is defined in (6.15)):

$$\begin{aligned} e(T, n) &= e_{\xi,\mathrm{RW}}(T, n) + e_{\eta,\mathrm{RW}}(T, n) + e_{\mathrm{br}}(T, n) \\ &=: \sum_{j=1}^{2} e_{\xi,\mathrm{RW},j}(T, n) + \sum_{j=1}^{2} e_{\eta,\mathrm{RW},j}(T, n) + e_{\mathrm{br}}(T, n), \end{aligned} \tag{6.19}$$

where

$$\begin{aligned} e_{\xi,\mathrm{RW},1}(T, n) = \mathrm{E}\int_0^{\beta_n(T)} \tilde{f}^n_{T,s}\Bigg\{\kappa \sum_{x,y\in\Lambda_n} p^n_{xy}\xi^n_s(x) \Big(2\Big(\tilde{u}^n_{\beta_n(T)-s}(x) - \tilde{u}^n_{\beta_n(T)-s}(y)\Big) \\ \times \Big(\tilde{v}^n_{\beta_n(T)-s}(x) - \tilde{v}^n_{\beta_n(T)-s}(y)\Big) \\ + i\Big[\Big(\tilde{u}^n_{\beta_n(T)-s}(x) - \tilde{u}^n_{\beta_n(T)-s}(y)\Big)^2 - \Big(\tilde{v}^n_{\beta_n(T)-s}(x) - \tilde{v}^n_{\beta_n(T)-s}(y)\Big)^2\Big]\Big)\Bigg\}\,\mathrm{d}s, \end{aligned}$$

$$e_{\xi,\mathrm{RW},2}(T, n) = \mathrm{E}\int_0^{\beta_n(T)} \tilde{f}^n_{T,s}\Bigg\{\kappa \sum_{x,y\in\Lambda_n} p^n_{xy}\xi^n_s(x) G^{1,1}\Big(\tilde{u}^n_{\beta_n(T)-s}, \tilde{v}^n_{\beta_n(T)-s}, x, y\Big)\Bigg\}\,\mathrm{d}s,$$

$$\begin{aligned} e_{\eta,\mathrm{RW},1}(T, n) = \mathrm{E}\int_0^{\beta_n(T)} \tilde{f}^n_{T,s}\Bigg\{\kappa \sum_{x,y\in\Lambda_n} p^n_{xy}\eta^n_s(x) \Big(2\,\Big(\tilde{u}^n_{\beta_n(T)-s}(x) - \tilde{u}^n_{\beta_n(T)-s}(y)\Big) \\ \times \Big(\tilde{v}^n_{\beta_n(T)-s}(x) - \tilde{v}^n_{\beta_n(T)-s}(y)\Big) \\ - \mathrm{i}\Big[\Big(\tilde{u}^n_{\beta_n(T)-s}(x) - \tilde{u}^n_{\beta_n(T)-s}(y)\Big)^2 - \Big(\tilde{v}^n_{\beta_n(T)-s}(x) - \tilde{v}^n_{\beta_n(T)-s}(y)\Big)^2\Big]\Big)\Bigg\}\,\mathrm{d}s, \end{aligned}$$

$$e_{\eta,\mathrm{RW},2}(T, n) = \mathrm{E}\int_0^{\beta_n(T)} \tilde{f}^n_{T,s}\Bigg\{\kappa \sum_{x,y\in\Lambda_n} p^n_{xy}\eta^n_s(x) G^{1,2}\Big(\tilde{u}^n_{\beta_n(T)-s}, \tilde{v}^n_{\beta_n(T)-s}, x, y\Big)\Bigg\}\,\mathrm{d}s,$$

$$e_{\mathrm{br}}(T, n) = \mathrm{E}\int_0^{\beta_n(T)} \tilde{f}^n_{T,s} \sum_{x\in\Lambda_n} \gamma\,\xi^n_s(x)\eta^n_s(x)\Bigg(\sum_{j=1}^{2} G^{2,j}(\tilde{u}^n_{\beta_n(T)-s}, \tilde{v}^n_{\beta_n(T)-s}, x)\Bigg)\,\mathrm{d}s.$$

Now we are going to show that indeed $e(T, n)$ vanishes, as $n \to \infty$. We start with the following technical lemma that was proved in Lemma 2.1 in [7].

Lemma 6.7. *Denote by $\{p^n_t(x, y) : t \geq 0, x, y \in \Lambda_n\}$ the transition probabilities of the symmetric nearest-neighbor random walk on the domain Λ_n, and let $\{p_t(x, y) : t \geq 0, x, y \in \mathbb{Z}^d\}$ denote the corresponding transition probabilities on $\mathbb{Z}^d$. Let $\{g_t(\cdot)\}_{t\geq 0}$ be the Green function of the symmetric nearest-neighbor random walk on $\mathbb{Z}^d$. Then the following hold.*

(a) *If $t_n/n^2 \to \infty$ as $n \to \infty$, then*

$$\sup_{t\geq t_n}\sup_{x,y\in\Lambda_n} (2n)^d\big|p^n_t(x, y) - (2n)^{-d}\big| \to 0.$$

(b) *If* $d \geq 3$, *and* $T(n)/|\Lambda_n| \to s \in (0, \infty)$ *as* $n \to \infty$, *then*

$$\lim_{n\to\infty} \int_0^{T(n)} p^n_{2t}(x, y)\,\mathrm{d}t = \int_0^{\infty} p_{2t}(x, y)\,\mathrm{d}t + s = \frac{1}{2} g_\infty(x-y) + s.$$

First we state the lemma that gives us an important bound on moments of the processes u^n, v^n. This is where the condition (2.8) on $\tilde{\gamma} = \gamma\sigma^2$ is used.

Lemma 6.8. *Let* $d \geq 3$ *and* $\tilde{\gamma} = \gamma\sigma^2 < \frac{1}{\sqrt{3^5}(\frac{1}{2}g_\infty(0)+1)}$. *Let* $\left(u^n_t, v^n_t\right)_{t\geq 0}$ *be a solution of* (1.3), *with* Q^n *being a* Q*-matrix of the nearest-neighbor random walk on* Λ_n. *Let* $\vartheta_1, \vartheta_2 \geq 0$. *Assume that* $u^n_0(x) = \vartheta_1$, $v^n_0(x) = \vartheta_2$ *for all* $x \in \Lambda_n$. *Then, for any* $T \leq 1$,

$$\sup_{n\geq 1}\ \sup_{0\leq t\leq \beta_n(T)}\ \sup_{x\in\Lambda_n} \mathrm{E}\left(\left(u^n_t(x)\right)^4\right) < \infty, \quad \text{and} \quad \sup_{n\geq 1}\ \sup_{0\leq t\leq \beta_n(T)}\ \sup_{x\in\Lambda_n} \mathrm{E}\left(\left(v^n_t(x)\right)^4\right) < \infty.$$

Proof. The proof is technical, however it follows easily from the proof of Lemma 2.2 in [7]. Since u^n, v^n have constant initial conditions it is easy to see that $\mathrm{E}(u^n_t(x)^p)$, $\mathrm{E}(v^n_t(x)^p)$, $\mathrm{E}(u^n_t(x)^p v^n_t(x)^p)$ are constant functions in x for $p > 0$. Thus, denote $f^n_t = \mathrm{E}((u^n_t(0))^4)$, $r^n_t = \mathrm{E}((v^n_t(0))^4)$, $t \geq 0$. Then following the argument in the proof of Lemma 2.2 in [7, pp.175–176], one gets that there exists a constant $C > 0$ such that

$$\begin{aligned} f^n(t) &\leq C\vartheta_1^4 + 3^5 \int_0^t p^n_{2s}(0)\,\mathrm{d}s \int_0^t p^n_{2s}(0)\tilde{\gamma}^2 \mathrm{E}\left(\left(u^n_{t-s}(0)v^n_{t-s}(0)\right)^2\right)\mathrm{d}s \\ &\leq C\vartheta_1^4 + 3^5 \int_0^t p^n_{2s}(0)\,\mathrm{d}s \int_0^t p^n_{2s}(0)\tilde{\gamma}^2 \frac{1}{2}(f^n(t-s) + r^n(t-s))\,\mathrm{d}s, \end{aligned} \tag{6.20}$$

and, similarly,

$$r^n(t) \leq C\vartheta_2^4 + 3^5 \int_0^t p^n_{2s}(0)\,\mathrm{d}s \int_0^t p^n_{2s}(0)\tilde{\gamma}^2 \frac{1}{2}(f^n(t-s) + r^n(t-s))\,\mathrm{d}s,$$

Letting $J^n(t) = \int_0^t p^n_{2s}(0)\,\mathrm{d}s$, and

$$\bar{h}^n_t = \sup_{s\leq t} f^n_s + \sup_{s\leq t} r^n_s, \quad t \geq 0,$$

we have

$$\bar{h}^n(t) \leq C\vartheta_1^4\vartheta_2^4 + 3^5 J^n(t)^2 \tilde{\gamma}^2 \bar{h}^n(t), \quad t \geq 0.$$

From Lemma 6.7 (b) we get that

$$\lim_{n\to\infty} J^n(\beta_n(t)) = \frac{1}{2} g_\infty(0) + t \leq \frac{1}{2} g_\infty(0) + 1, \quad \text{for } t \leq T \leq 1.$$

Recalling that $\tilde{\gamma} < \frac{1}{\sqrt{3^5}(\frac{1}{2}g_\infty(0)+1)}$ we get that

$$\limsup_{n\to\infty} \bar{h}^n(\beta_n(T)) \leq \frac{C\vartheta_1^4\vartheta_2^4}{1 - 3^5\tilde{\gamma}^2(\frac{1}{2}g_\infty(0)+1)^2} < \infty, \quad T \leq 1.$$

Since $\bar{h}^n(\beta_n(T)) < \infty$ for each finite n, we are done. □

From this, we derive the following corollary.

Corollary 6.1. *For any* $x, y \in \Lambda_n$,

$$\sup_n \sup_{t\le\beta_n(T)} \sup_{x,y\in\Lambda_n} \mathrm{E}\left(\xi_t^n(x)\xi_t^n(y)\right), \quad \sup_n \sup_{t\le\beta_n(T)} \sup_{x,y\in\Lambda_n} \mathrm{E}\left(\eta_t^n(x)\eta_t^n(y)\right) < \infty.$$

Proof. By Lemmas 5.3 and 5.4 it is enough to show that.

$$\sup_n \sup_{t\le\beta_n(T)} \sup_{x,y\in\Lambda_n} g_t^n(x, y) < \infty,$$

where $\left\{g_t^n(\cdot,\cdot)\right\}_{t\ge 0}$ is the Green function of the symmetric nearest-neighbor random walk on Λ_n. For any $t \ge 0$, $x, y \in \Lambda_n$, we have

$$g_t^n(x, y) \le g_{\beta_n(T)}^n(x, y) \le g_{\beta_n(T)}^n(0, 0).$$

By Lemma 6.7 (b) $\sup_n g_{\beta_n(T)}^n(0, 0)$ is finite, and we are done. □

Since, for $x, y \in \Lambda_n$, $p_t^n(x, y)$, $g_t^n(x, y)$ are functions of $x - y$, with some abuse of notation we will sometimes use the notation $p_t^n(x - y)$, $g_t^n(x - y)$ for $p_t^n(x, y)$, $g_t^n(x, y)$, respectively.

In what follows, we always assume that $\tilde{\gamma} = \gamma\sigma^2 < \frac{1}{\sqrt{3^5}(\frac{1}{2}g_\infty(0)+1)}$. With Lemma 6.8 at hand we are ready to treat the terms $e_{\mathrm{br}}(T, n)$, $e_{\xi,\mathrm{RW},2}(T, n)$, and $e_{\eta,\mathrm{RW},2}(T, n)$.

Lemma 6.9. *We have*

$$e_{\mathrm{br}}(T, n) \longrightarrow 0, \quad \text{as } n \to \infty, \tag{6.21}$$

$$e_{\xi,\mathrm{RW},2}(T, n) \longrightarrow 0, \quad \text{and} \quad e_{\eta,\mathrm{RW},2}(T, n) \longrightarrow 0, \quad \text{as } n \to \infty. \tag{6.22}$$

Proof. We will show only (6.21), since the proof of (6.22) follows along the similar lines:

$$\begin{aligned}
|e_{\mathrm{br}}(T, n)| &= \left|\mathrm{E}\int_0^{\beta_n(T)} \tilde{f}_{T,s}^n \sum_{x\in\Lambda_n} \gamma\xi_s^n(x)\eta_s^n(x)\left(\sum_{j=1}^2 G^{2,j}(\tilde{u}_{\beta_n(T)-s}^n, \tilde{v}_{\beta_n(T)-s}^n, x)\right) \mathrm{d}s\right| \\
&\le C_{(6.17)}\mathrm{E}\int_0^{\beta_n(T)} |\tilde{f}_{T,s}^n| \sum_{x\in\Lambda_n} \gamma\xi_s^n(x)\eta_s^n(x)\mathrm{e}^{\tilde{u}_{\beta_n(T)-s}^n(x)+\tilde{v}_{\beta_n(T)-s}^n(x)} \\
&\quad \times \left(\tilde{u}_{\beta_n(T)-s}^n(x)^3 + \tilde{v}_{\beta_n(T)-s}^n(x)^3\right) \mathrm{d}s \\
&\le C_{(6.17)}\gamma\mathrm{E}\int_0^{\beta_n(T)} \sum_{x\in\Lambda_n} \xi_s^n(x)\eta_s^n(x) \left(\tilde{u}_{\beta_n(T)-s}^n(x)^3 + \tilde{v}_{\beta_n(T)-s}^n(x)^3\right) \mathrm{d}s,
\end{aligned} \tag{6.23}$$

where the first inequality follows by (6.17) and the second inequality follows by the trivial inequality

$$|\tilde{f}_{T,s}^n|\xi_s^n(x)\eta_s^n(x)\mathrm{e}^{\tilde{u}_{\beta_n(T)-s}^n(x)+\tilde{v}_{\beta_n(T)-s}^n(x)} \le \xi_s^n(x)\eta_s^n(x),$$

for all $x \in \Lambda_n$ (recall the definition of $\tilde{f}_{T,s}^n$ in (6.18)).

Consider the process $(\hat{u}^n, \hat{v}^n)$ that solves (1.3) equations with initial conditions

$$\hat{u}_0^n(x) = |\Lambda_n|\tilde{u}_0^n(x), \quad \hat{v}_0^n(x) = |\Lambda_n|\tilde{v}_0^n(x), \quad \forall\, x \in \Lambda_n.$$

Then for any $s > 0$,

$$\hat{u}^n_s(x) = |\Lambda_n|\tilde{u}^n_s(x), \qquad \hat{v}^n_s(x) = |\Lambda_n|\tilde{v}^n_s(x), \qquad \forall\, x \in \Lambda_n.$$

Therefore, by the above, (6.23), and Fubini's theorem, we get

$$\begin{aligned}
|e_{\mathrm{br}}(T,n)| &\le |\Lambda_n|^{-3} C_{(6.17)}\gamma \int_0^{\beta_n(T)} \mathrm{E}\Bigg[\sum_{x\in\Lambda_n} \xi^n_s(x)\eta^n_s(x)\left(\hat{u}^n_{\beta_n(T)-s}(x)^3 + \hat{v}^n_{\beta_n(T)-s}(x)^3\right)\Bigg]\mathrm{d}s \\
&\le C_{(6.17)}\gamma T|\Lambda_n|^{-1}\theta_1\theta_2 \sup_{s\le\beta_n(T)} \frac{1}{|\Lambda_n|}\mathrm{E}\Bigg[\sum_{x\in\Lambda_n}\left(\left(\hat{u}^n_{\beta_n(T)-s}(x)\right)^3 + \left(\hat{v}^n_{\beta_n(T)-s}(x)\right)^3\right)\Bigg] \\
&\le C_{(6.17)}\gamma T|\Lambda_n|^{-1}\theta_1\theta_2 \sup_{x\in\Lambda_n}\sup_{s\le\beta_n(T)} \mathrm{E}\left[\left(\left(\hat{u}^n_{\beta_n(T)-s}(x)\right)^3 + \left(\hat{v}^n_{\beta_n(T)-s}(x)\right)^3\right)\right],
\end{aligned}$$

where the second inequality follows by Corollary 5.2, and the third inequality is trivial. With this, to obtain (6.21), it is enough to show that

$$\sup_{n\ge1}\sup_{x\in\Lambda_n}\sup_{s\le\beta_n(T)} \mathrm{E}\left[\left(\left(\hat{u}^n_{\beta_n(T)-s}(x)\right)^3 + \left(\hat{v}^n_{\beta_n(T)-s}(x)\right)^3\right)\right] < \infty.$$

However, this follows from Lemma 6.8 and Jensen's inequality. □

Before we begin analyzing the limiting behavior of $e_{\xi,\mathrm{RW},1}(T,n)$ and $e_{\eta,\mathrm{RW},1}(T,n)$, we require a technical lemma whose proof is simple and thus is omitted.

Lemma 6.10. *For any $n \in \mathbb{N}$, $r > 0$,*

$$\sum_{x_1,y_1\in\Lambda_n}\sum_{x_2,y_2\in\Lambda_n} p^n_{x_1,y_1}p^n_{x_2,y_2}\left(p^n_r(x_1-x_2) + p^n_r(y_1-y_2) + p^n_r(x_1-y_2) + p^n_r(y_1-x_2)\right) = 4\,|\Lambda_n|\,.$$

Now we are ready to derive the limiting behavior of $e_{\xi,\mathrm{RW},1}(T,n)$ and $e_{\eta,\mathrm{RW},1}(T,n)$.

Lemma 6.11. *We have*

$$\lim_{n\to\infty} e_{\xi,\mathrm{RW},1}(T,n) = 0, \qquad \lim_{n\to\infty} e_{\eta,\mathrm{RW},1}(T,n) = 0.$$

Proof. We will take care of $e_{\xi,\mathrm{RW},1}(T,n)$; the proof for $e_{\eta,\mathrm{RW},1}(T,n)$ is the same:

$$\begin{aligned}
&\left|e_{\xi,\mathrm{RW},1}(T,n)\right| \\
&\quad\le C\kappa\mathrm{E}\Bigg(\int_0^{\beta_n(T)}\left|\tilde{f}^n_{T,s}\right|\Bigg\{\Bigg|\sum_{x,y\in\Lambda_n} p^n_{xy}\xi^n_{\beta_n(T)-s}(x)\left(\tilde{u}^n_s(x)-\tilde{u}^n_s(y)\right)\left(\tilde{v}^n_s(x)-\tilde{v}^n_s(y)\right)\Bigg| \\
&\qquad + \Bigg|\sum_{x,y\in\Lambda_n} p^n_{xy}\xi^n_{\beta_n(T)-s}(x)\left(\tilde{u}^n_s(x)-\tilde{u}^n_s(y)\right)^2 - \left(\tilde{v}^n_s(x)-\tilde{v}^n_s(y)\right)^2\Bigg|\,\mathrm{d}s\Bigg\}\Bigg) \qquad (6.24) \\
&\quad\le C\mathrm{E}\left(\int_0^{\beta_n(T)} J^n_1(s)\,\mathrm{d}s + \int_0^{\beta_n(T)} J^n_2(s)\,\mathrm{d}s\right),
\end{aligned}$$

where

$$J_1^n(s) = \left| \sum_{x,y\in\Lambda_n} p_{xy}^n \xi_{\beta_n(T)-s}^n(x) \left(\tilde{u}_s^n(x) - \tilde{u}_s^n(y)\right) \left(\tilde{v}_s^n(x) - \tilde{v}_s^n(y)\right) \right|,$$

$$J_2^n(s) = \left| \sum_{x,y\in\Lambda_n} p_{xy}^n \xi_{\beta_n(T)-s}^n(x) \left(\left(\tilde{u}_s^n(x) - \tilde{u}_s^n(y)\right)^2 - \left(\tilde{v}_s^n(x) - \tilde{v}_s^n(y)\right)^2 \right) \right|.$$

Let us bound the expected value of J_1^n:

$$\begin{aligned} \mathrm{E}(J_1^n(s)) &= \mathrm{E}\left| \sum_{x,y\in\Lambda_n} p_{xy}^n \xi_{\beta_n(T)-s}^n(x) \left(\tilde{u}_s^n(x) - \tilde{u}_s^n(y)\right) \left(\tilde{v}_s^n(x) - \tilde{v}_s^n(y)\right) \right| \\ &\le \sqrt{\mathrm{E}\left[\left(\sum_{x,y\in\Lambda_n} p_{xy}^n \xi_{\beta_n(T)-s}^n(x) \left(\tilde{u}_s^n(x) - \tilde{u}_s^n(y)\right) \left(\tilde{v}_s^n(x) - \tilde{v}_s^n(y)\right) \right)^2 \right]}. \end{aligned}$$

Now we will recall the following representation from [16, Theorem 2.2] with $\varphi = \delta_x$:

$$\begin{cases} \tilde{u}_t^n(x) = P_t^n \tilde{u}_0^n(x) + \sum_{z\in\Lambda_n} \int_0^t p_{t-s}^n(x-z) \sqrt{\tilde{\gamma}\, \tilde{u}_s^n(z) \tilde{v}_s^n(z)}\, \mathrm{d}B_s(z), & x \in \Lambda_n, \\ \tilde{v}_t^n(x) = P_t^n \tilde{v}_0^n(x) + \sum_{z\in\Lambda_n} \int_0^t p_{t-s}^n(x-z) \sqrt{\tilde{\gamma}\, \tilde{u}_s^n(z) \tilde{v}_s^n(z)}\, \mathrm{d}W_s(z), & x \in \Lambda_n, \end{cases}$$

to get

$$\begin{aligned} N_r^t(x, y) &:= P_{t-r}^n \tilde{u}_r^n(x) - P_{t-r}^n \tilde{u}_r^n(y) \\ &= P_t^n \tilde{u}_0^n(x) - P_t^n \tilde{u}_0^n(y) \\ &\quad + \sum_{z\in\Lambda_n} \int_0^r \left(p_{t-s}^n(x-z) - p_{t-s}^n(y-z)\right) \sqrt{\tilde{\gamma}\, \tilde{u}_s^n(z) \tilde{v}_s^n(z)}\, \mathrm{d}B_s(z), \qquad r \le t, \end{aligned} \tag{6.25}$$

where the last equality follows from the Chapman–Kolmogorov formula. Similarly, for $r \le t$ we get

$$\begin{aligned} M_r^t(x, y) &:= P_{t-r}^n \tilde{v}_t^n(x) - P_{t-r}^n \tilde{v}_t^n(y) \\ &= P_t^n \tilde{v}_0^n(x) - P_t^n \tilde{v}_0^n(y) \\ &\quad + \sum_{z\in\Lambda_n} \int_0^r \left(p_{t-s}^n(x-z) - p_{t-s}^n(y-z)\right) \sqrt{\tilde{\gamma}\, \tilde{u}_s^n(z) \tilde{v}_s^n(z)}\, \mathrm{d}W_s(z) \end{aligned} \tag{6.26}$$

where $\{B_\cdot(z)\}_{z\in\Lambda_n}$, $\{W_\cdot(z)\}_{z\in\Lambda_n}$ are orthogonal Brownian motions.

Note that $\left\{N_r^t(x, y)\right\}_{0\le r\le t}$ and $\left\{M_r^t(x, y)\right\}_{0\le r\le t}$ are martingales; in addition

$$\begin{aligned} \tilde{u}_t^n(x) - \tilde{u}_t^n(y) &= P_{t-r}^n \tilde{u}_r^n(x) - P_{t-r}^n \tilde{u}_r^n(y)\Big|_{r=t} = N_r^t(x, y)\Big|_{r=t}, \\ \tilde{v}_t^n(x) - \tilde{v}_t^n(y) &= P_{t-r}^n \tilde{v}_r^n(x) - P_{t-r}^n \tilde{v}_r^n(y)\Big|_{r=t} = M_r^t(x, y)\Big|_{r=t}. \end{aligned} \tag{6.27}$$

Then by orthogonality of the Brownian motions $B_\cdot(z)$ and $W_\cdot(z)$ for all $z\in\Lambda_n$, and the Itô formula we get

$$
\begin{aligned}
&\sum_{x,y\in\Lambda_n} p^n_{x,y}\xi^n_{\beta_n(T)-s} M^s_s(x,y)N^s_s(x,y)\\
&\qquad = \sum_{x,y\in\Lambda_n} p^n_{xy}\xi^n_{\beta_n(T)-s}(x)\left(\tilde u^n_s(x)-\tilde u^n_s(y)\right)\left(\tilde v^n_s(x)-\tilde v^n_s(y)\right)\\
&\qquad = \sum_{x,y\in\Lambda_n} p^n_{xy}\xi^n_{\beta_n(T)-s}(x)\sum_{z\in\Lambda_n}\int_0^s \left(P^n_{s-r}\tilde u^n_r(x)-P^n_{s-r}\tilde u^n_r(y)\right)\\
&\qquad\quad \times \left(p^n_{s-r}(x-z)-p^n_{s-r}(y-z)\right)\sqrt{\tilde\gamma\,\tilde u^n_r(z)\tilde v^n_r(z)}\,\mathrm{d}W_r(z)\\
&\qquad\quad + \sum_{x,y\in\Lambda_n} p^n_{xy}\xi^n_{\beta_n(T)-s}(x)\sum_{z\in\Lambda_n}\int_0^s \left(P^n_{s-r}\tilde v^n_r(x)-P^n_{s-r}\tilde v^n_r(y)\right)\\
&\qquad\quad \times \left(p^n_{s-r}(x-z)-p^n_{s-r}(y-z)\right)\sqrt{\tilde\gamma\,\tilde u^n_r(z)\tilde v^n_r(z)}\,\mathrm{d}B_r(z)\\
&\qquad =: \sum_{x,y\in\Lambda_n} p^n_{xy}\xi^n_{\beta_n(T)-s}(x)\sum_{z\in\Lambda_n}\tilde I^n_{1,1}(s,x,y,z)\\
&\qquad\quad + \sum_{x,y\in\Lambda_n} p^n_{xy}\xi^n_{\beta_n(T)-s}(x)\sum_{z\in\Lambda_n}\tilde I^n_{1,2}(s,x,y,z)\\
&\qquad =: I^n_{1,1}(s)+I^n_{1,2}(s).
\end{aligned}
$$

Note that

$$
\mathrm{E}\left(J^n_1(s)\right)\le C\sqrt{\left(\mathrm{E}\left[I^n_{1,1}(s)\right)^2\right]+\mathrm{E}\left[\left(I^n_{1,2}(s)\right)^2\right]}. \tag{6.28}
$$

Thus, let us bound $\mathrm{E}[(I^n_{1,1}(s))^2]$: for all $s\le\beta_n(T)$, we have

$$
\begin{aligned}
\mathrm{E}\left[\left(I^n_{1,1}(s)\right)^2\right] = &\sum_{x_1,y_1\in\Lambda_n}\sum_{x_2,y_2\in\Lambda_n}\mathrm{E}\left(\xi^n_{\beta_n(T)-s}(x_1)\xi^n_{\beta_n(T)-s}(x_2)\right)\\
&\times p^n_{x_1,y_1}p^n_{x_2,y_2}\sum_{z_1\in\Lambda_n}\sum_{z_2\in\Lambda_n}\mathrm{E}\left[\tilde I^n_{1,1}(s,x_1,y_1,z_1)\tilde I^n_{1,1}(s,x_2,y_2,z_2)\right].
\end{aligned} \tag{6.29}
$$

Note that for $z_1\neq z_2$, $\tilde I^n_{1,1}(r,x_1,y_1,z_1)$ and $\tilde I^n_{1,1}(r,x_2,y_2,z_2)$ are orthogonal square integrable martingales for $r\le s$ and, hence,

$$
\begin{aligned}
&\sum_{z_1\in\Lambda_n}\sum_{z_2\in\Lambda_n}\mathrm{E}\left[\tilde I^n_{1,1}(s,x_1,y_1,z_1)\tilde I^n_{1,1}(s,x_2,y_2,z_2)\right]\\
&\quad = \sum_{z\in\Lambda_n}\mathrm{E}\left[\tilde I^n_{1,1}(s,x_1,y_1,z)\tilde I^n_{1,1}(s,x_2,y_2,z)\right]\\
&\quad = \tilde\gamma\sum_{z\in\Lambda_n}\mathrm{E}\Bigg[\int_0^s \left(P^n_{s-r}\tilde u^n_r(x_1)-P^n_{s-r}\tilde u^n_r(y_1)\right)\left(p^n_{s-r}(x_1-z)-p^n_{s-r}(y_1-z)\right)\\
&\qquad \times \left(P^n_{s-r}\tilde u^n_r(x_2)-P^n_{s-r}\tilde u^n_r(y_2)\right)\left(p^n_{s-r}(x_2-z)-p^n_{s-r}(y_2-z)\right)\tilde u^n_r(z)\tilde v^n_r(z)\,\mathrm{d}r\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&= \tilde{\gamma} \sum_{z\in\Lambda_n} \mathrm{E}\Bigg[\int_0^s \sum_{z_1\in\Lambda_n} \left(p^n_{s-r}(x_1-z_1)-p^n_{s-r}(y_1-z_1)\right)\tilde{u}^n_r(z_1) \\
&\quad \times \sum_{z_2\in\Lambda_n} \left(p^n_{s-r}(x_2-z_2)-p^n_{s-r}(y_2-z_2)\right)\tilde{u}^n_r(z_2) \\
&\quad \times \left(p^n_{s-r}(x_1-z)-p^n_{s-r}(y_1-z)\right)\left(p^n_{s-r}(x_2-z)-p^n_{s-r}(y_2-z)\right) \\
&\quad \times \tilde{u}^n_r(z)\tilde{v}^n_r(z)\,\mathrm{d}r\Bigg] \\
&\le \tilde{\gamma} \sum_{z\in\Lambda_n} \int_0^s \sum_{z_1\in\Lambda_n}\sum_{z_2\in\Lambda_n} \hat{J}_{1,1}(\vec{x},\vec{y},\vec{z},s-r)\mathrm{E}\left[\tilde{u}^n_r(z_1)\tilde{u}^n_r(z_2)\tilde{u}^n_r(z)\tilde{v}^n_r(z)\right]\,\mathrm{d}r,
\end{aligned}
$$

where $\vec{x}=(x_1,x_2)$, $\vec{y}=(y_1,y_2)$, $\vec{z}=(z_1,z_2,z)$ and

$$
\hat{J}_{1,1}(\vec{x},\vec{y},\vec{z},s-r)=\prod_{i=1,2}\left|p^n_{s-r}(x_i-z_i)-p^n_{s-r}(y_i-z_i)\right|\left|p^n_{s-r}(x_i-z)-p^n_{s-r}(y_i-z)\right|.
$$

By Lemma 6.8 and assumption on the initial conditions of $(\tilde{u},\tilde{v})$,

$$
\mathrm{E}\left[\tilde{u}^n_r(z_1)\tilde{u}^n_r(z_2)\tilde{u}^n_r(z)\tilde{v}^n_r(z)\right]
$$

is bounded by $C\,|\Lambda_n|^{-4}$ uniformly on $z, z_1, z_2\in\Lambda_n$, $r\le\beta_n(T)$ and $n\ge 1$. Therefore,

$$
\begin{aligned}
&\sum_{z_1\in\Lambda_n}\sum_{z_2\in\Lambda_n}\mathrm{E}\left[\tilde{I}^n_{1,1}(s,x_1,y_1,z_1)\tilde{I}^n_{1,1}(s,x_2,y_2,z_2)\right] \\
&\qquad \le C\,|\Lambda_n|^{-4}\sum_{z\in\Lambda_n}\int_0^s\sum_{z_1\in\Lambda_n}\sum_{z_2\in\Lambda_n}\hat{J}_{1,1}(\vec{x},\vec{y},\vec{z},s-r)\,\mathrm{d}r. \qquad (6.30)
\end{aligned}
$$

Denote

$$
\tilde{J}_{1,1}(\vec{x},\vec{y},s-r):=\sum_{z\in\Lambda_n}\sum_{z_1\in\Lambda_n}\sum_{z_2\in\Lambda_n}\hat{J}_{1,1}(\vec{x},\vec{y},\vec{z},s-r).
$$

Now we decompose the term on the right-hand side of (6.30) into two terms

$$
C\,|\Lambda_n|^{-4}\int_0^{(s-n^\delta)_+}\tilde{J}_{1,1}(\vec{x},\vec{y},s-r)dr + C\,|\Lambda_n|^{-4}\int_{(s-n^\delta)_+}^{s}\tilde{J}_{1,1}(\vec{x},\vec{y},s-r)dr \qquad (6.31)
$$

for some $\delta\in(2,d)$.

By Lemma 6.7 (a) we get

$$
\lim_{n\to\infty}\sup_{t>n^\delta}\sup_{z_1,z_2\in\Lambda_n}|\Lambda_n|\left|p^n_t(z_1,z_2)-(2n)^{-d}\right|=0,
$$

for any $\delta>2$. This implies that, for any $\delta>2$, there exists a sequence $a_n=a_n(\delta)$, such that

$$
\sup_{s\ge n^\delta}\sup_{w_1,w_2,w_3\in\Lambda_n}\left|p^n_s(w_1,w_2)-p^n_s(w_3,w_2)\right|\le\frac{a_n}{|\Lambda_n|}, \qquad (6.32)
$$

where $a_n\to 0$, as $n\to\infty$.

By (6.32) we immediately get

$$\tilde{J}_{1,1}(\vec{x}, \vec{y}, s-r) \le |\Lambda_n|^3\, a_n^4\, |\Lambda_n|^{-4} = |\Lambda_n|^{-1}\, a_n^4, \tag{6.33}$$

for $s > n^\delta$ and $r \le s - n^\delta$. Hence for $s \le \beta_n(T)$, we get

$$|\Lambda_n|^{-4} \int_0^{(s-n^\delta)_+} \tilde{J}_{1,1}(\vec{x}, \vec{y}, s-r)dr \le |\Lambda_n|^{-4}\, a_n^4\, |\Lambda_n|^{-1} \int_0^{(s-n^\delta)_+} 1dr \le C\, |\Lambda_n|^{-4}\, a_n^4, \tag{6.34}$$

where the last inequality follows since $s \le \beta_n(T) = |\Lambda_n|\, T$.

Let us treat the second term in (6.31). Note that

$$\sum_{z_i \in \Lambda_n} \left|p_s^n(x_i - z_i) - p_s^n(y_i - z_i)\right| \le \sum_{z_i \in \Lambda_n} p_s^n(x_i - z_i) + p_s^n(y_i - z_i) \le 2, \forall\, i = 1, 2,\ \ s \ge 0.$$

In addition,

$$\begin{aligned}
&\sum_{z \in \Lambda_n} \left|p_{s-r}^n(x_1 - z) - p_{s-r}^n(y_1 - z)\right| \left|p_{s-r}^n(x_2 - z) - p_{s-r}^n(y_2 - z)\right| \\
&\qquad \le \sum_{z \in \Lambda_n} \left(p_{s-r}^n(x_1 - z) + p_{s-r}^n(y_1 - z)\right)\left(p_{s-r}^n(x_2 - z) + p_{s-r}^n(y_2 - z)\right) \\
&\qquad = p_{2(s-r)}^n(x_1 - x_2) + p_{2(s-r)}^n(y_1 - y_2) + p_{2(s-r)}^n(x_1 - y_2) + p_{2(s-r)}^n(y_1 - x_2),
\end{aligned}$$

where the last equality follows from the Chapman–Kolmogorov formula. Then

$$\begin{aligned}
&C\, |\Lambda_n|^{-4} \int_{(s-n^\delta)_+}^{s} \tilde{J}_{1,1}(\vec{x}, \vec{y}, s-r)\, \mathrm{d}r \\
&\quad \le C\, |\Lambda_n|^{-4} \int_0^{n^\delta} \left(p_{2r}^n(x_1 - x_2) + p_{2r}^n(y_1 - y_2) + p_{2r}^n(x_1 - y_2) + p_{2r}^n(y_1 - x_2)\right)\, \mathrm{d}r.
\end{aligned}\tag{6.35}$$

By (6.30), (6.31), (6.34), and (6.35) we get

$$\begin{aligned}
&\sum_{z_1 \in \Lambda_n} \sum_{z_2 \in \Lambda_n} \mathrm{E}\left[\tilde{I}_{1,1}^n(s, x_1, y_1, z_1)\tilde{I}_{1,1}^n(s, x_2, y_2, z_2)\right] \\
&\le C\, |\Lambda_n|^{-4} \left(a_n^4 + \int_0^{n^\delta} \left(p_{2r}^n(x_1 - x_2) + p_{2r}^n(y_1 - y_2) + p_{2r}^n(x_1 - y_2) + p_{2r}^n(y_1 - x_2)\right) dr\right).
\end{aligned}$$

Use the above inequality, (6.29), and also Corollary 6.1 and Lemma 6.10 to get

$$\begin{aligned}
\mathrm{E}\left[\left(I_{1,1}^n(s)\right)^2\right] &\le C\Bigg(\sum_{x_1, y_1 \in \Lambda_n} \sum_{x_2, y_2 \in \Lambda_n} p_{x_1, y_1}^n p_{x_2, y_2}^n\, |\Lambda_n|^{-4}\, a_n^4 + |\Lambda_n|^{-4}\, n^\delta\, |\Lambda_n|\Bigg) \\
&\le C\big(\, |\Lambda_n|^{-2}\, a_n^4 + |\Lambda_n|^{-3}\, n^\delta\big).
\end{aligned}\tag{6.36}$$

In the same way, we handle $I^n_{1,2}(s)$ and get

$$\begin{aligned}\mathrm{E}\big[\big(I^n_{1,2}(s)\big)^2\big] &\le C\Big(\sum_{x_1,y_1\in\Lambda_n}\sum_{x_2,y_2\in\Lambda_n} p^n_{x_1,y_1}p^n_{x_2,y_2}\,|\Lambda_n|^{-4}\,a_n^4 + |\Lambda_n|^{-4}\,n^\delta\,|\Lambda_n|\Big)\\ &\le C\big(\,|\Lambda_n|^{-2}\,a_n^4 + |\Lambda_n|^{-3}\,n^\delta\big). \end{aligned} \tag{6.37}$$

By (6.36), (6.37), and (6.28), we have

$$\begin{aligned}\mathrm{E}\left[J^n_1(s)\right] &\le \sqrt{\mathrm{E}\Big[\Big(I^n_{1,1}(s)\Big)^2 + \Big(I^n_{1,2}(s)\Big)^2\Big]}\\ &\le C\sqrt{|\Lambda_n|^{-2}\,a_n^4 + |\Lambda_n|^{-3}\,n^\delta}\\ &\le C\,|\Lambda_n|^{-1}\,a_n^2 + C\,|\Lambda_n|^{-3/2}\,n^{\delta/2}.\end{aligned}$$

Thus,

$$\begin{aligned}\int_0^{\beta_n(T)} \mathrm{E}\left(J^n_1(s)\right)\,\mathrm{d}s &\le C\int_0^{\beta_n(T)}\left(|\Lambda_n|^{-1}\,a_n^2 + |\Lambda_n|^{-3/2}\,n^{\delta/2}\right)\mathrm{d}s \\ &\le C\,|\Lambda_n|\left(|\Lambda_n|^{-1}\,a_n^2 + |\Lambda_n|^{-3/2}\,n^{\delta/2}\right)\\ &\le Ca_n^2 + C\left(\frac{n^\delta}{|\Lambda_n|}\right)^{1/2} \to 0, \quad \text{as } n\to\infty,\end{aligned} \tag{6.38}$$

where the last convergence holds since $\delta < d$ and $|\Lambda_n| = (2n+1)^d$.

Now we are ready to treat $J^n_2(s)$ in a similar way.

By the Itô formula,

$$\big(M^s_r(x,y)\big)^2 = \int_0^s M^s_r(x,y)\,\mathrm{d}M^s_r(x,y) + \big\langle M^s_\cdot(x,y)\big\rangle_s$$

and

$$\big(N^s_r(x,y)\big)^2 = \int_0^s N^s_r(x,y)\,\mathrm{d}N^s_r(x,y) + \big\langle N^s_\cdot(x,y)\big\rangle_s\,.$$

Note that $\langle M^t_\cdot(x,y)\rangle_t = \langle N^t_\cdot(x,y)\rangle_t$, and recall (6.27); therefore

$$J^n_2(s) = \sum_{x,y\in\Lambda_n}\frac{1}{2}p^n_{x,y}\xi^n_{\beta_n(T)-s}\left[\int_0^s M^s_r(x,y)\,\mathrm{d}M^s_r(x,y) - \int_0^s N^s_r(x,y)\,\mathrm{d}N^s_r(x,y)\right].$$

If we follow the steps of computations for $J^n_1(s)$, we get that

$$\lim_{n\to\infty}\mathrm{E}\left(\int_0^{\beta_n(T)} J^n_2(s)\,\mathrm{d}s\right) = 0.$$

Use this, (6.38), and (6.24) to finish the proof. □

Proof of Proposition 6.3. Proposition 6.3 follows immediately from (6.19), and Lemmas 6.9 and 6.11.

Acknowledgments

LM is supported in part by ISF grant Nos. 1704/18 and 1985/22.

Funding information

There are no funding bodies to thank relating to the creation of this article.

Competing interests

There were no competing interests to declare which arose during the preparation or publication process of this article.

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