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A new generic vanishing theorem on homogeneous varieties and the positivity conjecture for triple intersections of Schubert cells

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Abstract

In this paper we prove a new generic vanishing theorem for X a complete homogeneous variety with respect to an action of a connected algebraic group. Let $A, B_0 \subset X$ be locally closed affine subvarieties, and assume that B_0 is smooth and pure dimensional. Let $\mathcal{P}$ be a perverse sheaf on A and let $B = gB_0$ be a generic translate of B_0. Then our theorem implies $(-1)^{\operatorname{codim} B}\chi(A \cap B, \mathcal{P}|_{A\cap B}) \geq 0$. As an application, we prove in full generality a positivity conjecture about the signed Euler characteristic of generic triple intersections of Schubert cells. Such Euler characteristics are known to be the structure constants for the multiplication of the Segre–Schwartz–MacPherson classes of these Schubert cells.

1. Introduction

Let G be a complex semisimple, simply connected, linear algebraic group, and fix a Borel subgroup B with a maximal torus $T \subseteq B$. Let B^- denote the opposite Borel subgroup, with $P \subset G$ a parabolic subgroup containing B. Let W be the Weyl group of B, and let $w_0 \in W$ be the longest element in W; then $B^- = w_0 B w_0$. Let W_P be the subgroup of W generated by the simple reflections in P and denote by $W^P \subset W$ the set of minimal length representatives for the cosets of W_P in W.

Given any element $\lambda \in W^P$, let X_λ° and $X^{\lambda\circ}$ be the corresponding Schubert cell and the opposite Schubert cell in the partial flag variety $X := G/P$, respectively. Denote by X_λ and X^λ the corresponding Schubert variety and opposite Schubert variety. The Schubert cells X_λ° (respectively, opposite Schubert cells $X^{\lambda\circ}$) are B-orbits (respectively, B^--orbits) for the left multiplication action on X. Moreover,

$$X_\lambda = \bigsqcup_{\xi\in W^P, \xi\leq\lambda} X_\xi^\circ \quad \text{and} \quad X^\nu = \bigsqcup_{\xi\in W^P, \xi\geq\nu} X^{\xi\circ}$$

are algebraic Whitney stratifications of the Schubert variety X_λ and the opposite Schubert variety X^ν, respectively. By a classical result of Richardson [Ric92], these stratifications of X_λ

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and X^ν are transversal in X, so that the Richardson variety $X_\lambda \cap X^\nu$ admits an induced algebraic Whitney stratification.

For a locally closed algebraic subset $U \subset X$, let

$$c_{SM}(U) := c_M(1_U) \quad \text{and} \quad s_{SM}(U) := \frac{c_{SM}(U)}{c(TX)} \in H^*(X, \mathbb{Z})$$

be the Chern–Schwartz–MacPherson (CSM) class and the Segre–Schwartz–MacPherson (SSM) class of U or 1_U in the ambient variety X, respectively. Then by an intersection formula of the first author [Sch17, Theorem 1.2], the transversality of these stratifications implies the formula

$$c_{SM}(X_\lambda^\circ) \cdot s_{SM}(X^{\nu\circ}) = c_{SM}(X_\lambda^\circ \cap X^{\nu\circ}) \in H^*(X, \mathbb{Z}),$$

with $X_\lambda^\circ \cap X^{\nu\circ}$ an open Richardson variety. By counting T-fixed points in $X_\lambda^\circ \cap X^{\nu\circ}$ and the functoriality of the MacPherson–Chern class transformation c_M, the above formula implies the geometric orthogonality relation (see [AMSS23, Theorem 7.1])

$$\langle c_{SM}(X_\lambda^\circ), s_{SM}(X^{\nu\circ})\rangle := \int_X c_{SM}(X_\lambda^\circ) \cdot s_{SM}(X^{\nu\circ}) = \chi(X_\lambda^\circ \cap X^{\nu\circ}) = \delta_{\lambda,\nu}. \tag{1}$$

For any $\lambda, \mu, \nu \in W^P$, we introduce the SSM structure constants $a_{\lambda,\mu}^\nu \in \mathbb{Z}$ via

$$s_{SM}(X_\lambda^\circ) \cdot s_{SM}(X_\mu^\circ) = \sum_\nu a_{\lambda,\mu}^\nu \cdot s_{SM}(X_\nu^\circ) \in H^*(X, \mathbb{Z}).$$

By Kleiman's transversality theorem [Kle74], for generic $g \in G$, the translate gX_μ is stratified transversal to the Richardson variety $X_\lambda \cap X^\nu$ (with its induced algebraic Whitney stratification). Then the intersection formula [Sch17, Theorem 1.2] together with the geometric orthogonality relation (1) implies (see, for example, [Su21, Kum22, AMSS22b])

$$a_{\lambda,\mu}^\nu = \chi(X_\lambda^\circ \cap gX_\mu^\circ \cap X^{\nu\circ}) = \chi(X_\lambda^\circ \cap gX_\mu^\circ \cap hX_{\nu'}^\circ),$$

for $g, h \in G$ generic and $\nu' \cdot W_P = w_0\nu \cdot W_P$. Note that $c_{SM}(gX_\mu^\circ) = c_{SM}(X_\mu^\circ)$ by functoriality of the MacPherson–Chern class transformation, together with the fact that the connected group G acts trivially on $H^*(X, \mathbb{Z})$. Similarly $c_{SM}(X^{\nu\circ}) = c_{SM}(hX_{\nu'}^\circ)$, since there is an element $n_0 \in G$ (lifting $w_0 \in W$) with $n_0 X_{\nu'}^\circ = X^{\nu\circ}$ (see, for example, [AMSS23, Equation (29)], also for the corresponding T-equivariant result). Several authors formulated the following conjecture, posed from 2019 onward in talks of L. Mihalcea and A. Knutson, then in [KZ21] and finally in [Kum22, Conjecture D], as surveyed in [AMSS22b, Conjecture 3].

CONJECTURE 1.1 (Signed Euler characteristic of the intersection of three Schubert cells).

$$E_{\lambda,\mu,\nu'} := (-1)^d \cdot a_{\lambda,\mu}^\nu = (-1)^d \cdot \chi(X_\lambda^\circ \cap gX_\mu^\circ \cap hX_{\nu'}^\circ) \geq 0,$$

for $g, h \in G$ generic, with $\nu' \cdot W_P = w_0\nu \cdot W_P$ and $d := \dim(X_\lambda^\circ \cap gX_\mu^\circ \cap hX_{\nu'}^\circ)$.

Remark 1.2. By [Su21, Theorem 2.5] or [AMSS23, Eq. (36)], up to a sign, $E_{\lambda,\mu,\nu'}$ are also the corresponding structure constants for the CSM classes of Schubert cells in a full flag variety G/B.

Kumar [Kum22, Theorem 16] showed, for the case of full flag varieties G/B, that this conjecture would follow from positivity of the Chern–Schwartz–MacPherson classes $c_{SM}(X_\lambda^\circ \cap X^{\nu\circ})$ of all open Richardson varieties in G/B, posited by [Kum22, Conjecture 5] and [FGX22, Conjecture 9.2]. In [KZ21, Theorem 3], Knutson and Zinn-Justin proved for the case of the

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r-step partial flag variety of type A, with $r \leq 4$, that the Euler characteristic

$$E_{\lambda,\mu,\nu'} := (-1)^d \cdot a^{\nu}_{\lambda,\mu} = (-1)^d \cdot \chi(X^\circ_\lambda \cap gX^\circ_\mu \cap hX^\circ_{\nu'})$$

is equal to a weighted count of certain combinatorial puzzles. In particular, when $r \leq 3$, they obtained the following theorem (see also the recent ICM talk of Knutson [Knu22, Theorem 4]):

THEOREM 1.3 [KZ21]. *If X°_λ, X°_μ, and $X^\circ_{\nu'}$ are three Schubert cells in an r-step partial flag variety of type A, with $r \leq 3$, then $E_{\lambda,\mu,\nu'} \geq 0$.*

In this paper, we prove the following new generic vanishing theorem on homogeneous varieties, from which we deduce Conjecture 1.1 in full generality for all partial flag varieties $X = G/P$ (with functors like $f_!, f_*$ already denoting their derived versions).

THEOREM 1.4 (New generic vanishing theorem). *Let X be a complete homogeneous variety with respect to an action of a connected algebraic group G'. Let $A, B_0 \subset X$ be locally closed affine subvarieties, and assume that B_0 is also smooth and pure dimensional. Let $\mathcal{P}$ be a perverse sheaf on A and $B = gB_0$ be a generic translate of B_0. Then*

$$H^i_c\big(B, j_{A*}(\mathcal{P}|_{A\cap B})\big) \cong H^i\big(A, j_{B!}(\mathcal{P}|_{A\cap B})\big) = 0 \quad \textit{for all } i \neq -\operatorname{codim}_X B,$$

where $j_B : A\cap B \to A$ and $j_A : A \cap B \to B$ are the inclusion maps.

Note that such a complete homogeneous variety X is isomorphic to a product $X \cong Ab \times G/P$ of an abelian variety Ab and a partial flag variety G/P as before (see, for example, [Bri12, Theorem 2.6]).

The following result is a straightforward consequence of Theorem 1.4.

COROLLARY 1.5. *In the notation of the theorem,*

$$(-1)^{\operatorname{codim} B}\chi(A\cap B, \mathcal{P}|_{A\cap B}) \geq 0.$$

Here we assume that $\mathcal{P}$ is a perverse sheaf of vector spaces, with finite dimensional stalks (over a given field $\mathbb{K}$, for example $\mathbb{K} = \mathbb{C}$, in which case $\mathcal{P}$ is isomorphic the de Rham complex of a holonomic D-module), so that the Euler characteristics

$$\chi\big(H^i_c\big(B, j_{A*}(\mathcal{P}|_{A\cap B})\big)\big) \quad \text{and} \quad \chi\big(H^i\big(A, j_{B!}(\mathcal{P}|_{A\cap B})\big)\big)$$

only depend on the associated constructible function χ_{stalk} (given by the stalkwise Euler characteristic):

$$\chi_{\mathrm{stalk}}\big(j_{A*}(\mathcal{P}|_{A\cap B})\big) = j_{A*}\chi_{\mathrm{stalk}}(\mathcal{P}|_{A\cap B}) \quad \text{and} \quad \chi_{\mathrm{stalk}}\big(j_{B!}(\mathcal{P}|_{A\cap B})\big) = j_{B!}\chi_{\mathrm{stalk}}(\mathcal{P}|_{A\cap B})\,,$$

with $f_! = f_*$ as covariant functors on the level of constructible functions for an algebraic morphism f (see [MS22, Equation (10.3) and Example 10.4.37] and [Sch03, §§ 2.3 and 6.0.6]). So when $f = j$ is a locally closed inclusion, $j_! = j_*$ on the level of constructible functions is just the extension by zero.

Conjecture 1.1 can be deduced from Corollary 1.5, with $X = G/P$ and $G' = G$ and the following choices: $A = X^\circ_\lambda \cap gX^\circ_\mu$, $B_0 = X^\circ_{\nu'}$ and $\mathcal{P} = \mathbb{K}_A[\dim A]$. The results of Theorem 1.4 and Corollary 1.5 can easily be inductively applied for n ($n \geq 2$) generic intersections of Schubert cells. Let $A = X^\circ_{\mu_1} \cap g_2X^\circ_{\mu_2} \cap \cdots \cap g_{n-1}X^\circ_{\mu_{n-1}}$, $B_0 = X^\circ_{\mu_n}$ and $\mathcal{P} = \mathbb{K}_A[\dim A]$, where $g_2, \ldots, g_{n-1} \in G$ are generic.

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THEOREM 1.6. *Let $X^\circ_{\mu_1}, \dots, X^\circ_{\mu_n}$ be n Schubert cells ($n \geq 2$) in a flag variety $X = G/P$. Then for generic $g_2, \dots, g_n \in G$ one has*

$$(-1)^d \chi(X^\circ_{\mu_1} \cap g_2 X^\circ_{\mu_2} \cap \cdots \cap g_n X^\circ_{\mu_n}) \geq 0,$$

with $d = \dim(X^\circ_{\mu_1} \cap g_2 X^\circ_{\mu_2} \cap \cdots \cap g_n X^\circ_{\mu_n})$.

Note that we need to intersect at least two generically translated affine Schubert cells, so that our method applies. Of course, $(-1)^{\dim X_\mu} \cdot \chi(X^\circ_\mu) = (-1)^{\dim X_\mu}$ is not always non-negative for one Schubert cell X°_μ. These iterated open intersections are the open counterparts of the intersection varieties studied in [BC12]. In particular, when $n = 2$, one gets $(-1)^d \chi(X^\circ_\lambda \cap gX^\circ_\mu) \geq 0$, with $d = \dim(X^\circ_\lambda \cap gX^\circ_\mu)$ and $g \in G$ generic. This is consistent with the well-known result that for generic g, $X^\circ_\lambda \cap gX^\circ_\mu$ is isomorphic to an open Richardson variety $X^\circ_\lambda \cap X^{\nu\circ}$ (see [BC12, Remark 2.2]), so that this signed Euler characteristic is one for $d = 0$ and zero otherwise (by the geometric orthogonality relation (1)).

In this paper, Theorem 1.4 is deduced from Artin's vanishing theorem for perverse sheaves (see [MS22, Theorem 10.3.59] and [Sch03, Corollary 6.0.4]), Kleiman's transversality theorem [Kle74], together with the following key technical proposition.

PROPOSITION 1.7 (Generic base change isomorphism). *Let X be a homogeneous variety with respect to an action of a connected algebraic group G'. Let A, B_0 be locally closed subvarieties of X. Let $\mathcal{F}$ be a constructible complex on A . Let $B = gB_0$ be a generic translate of B_0. Denote various inclusion maps as in the following diagram.*

$$\begin{tikzcd} A \cap B \arrow[r, "j_B"] \arrow[d, "j=j_A"] & A \arrow[d, "i_A"] \\ B \arrow[r, "i_B"] & X \end{tikzcd}$$

Then there exists a canonical base change isomorphism in $D^b_c(X)$,

$$i_{B!} j_{A*}(\mathcal{F}|_{A\cap B}) \cong i_{A*} j_{B!}(\mathcal{F}|_{A\cap B}). \tag{2}$$

Remark 1.8. When $X = \mathbb{P}^n$ and $A = B = \mathbb{C}^n$ being an affine chart of X, the above base change isomorphism specializes to a theorem of Beilinson [Bei78]. This result of Beilinson played a key role in the study of perverse filtrations by de Cataldo and Migliorini [dCM10]. In fact, they proved our base change isomorphism in the case when $X = \mathbb{P}^n$ and $B = \mathbb{C}^n$ (see [dCM10, p. 2101]).

Conventions

Throughout this paper, we fix a commutative ring $\mathbb{K}$ with the unit as the coefficient ring, so that every (bounded) constructible complex and perverse sheaf has coefficients over $\mathbb{K}$. When taking Euler characteristics, we assume $\mathbb{K}$ is a field and all stalks of our constructible sheaf complexes are finite dimensional over $\mathbb{K}$. All the pushforward and pullback functors are derived functors. We work over complex algebraic varieties, and all stratifications are assumed to be algebraic Whitney stratifications. Given finitely many bounded constructible complexes on a given complex variety, there is always such a stratification with all the sheaf complexes constructible with respect to this stratification, that is, all their cohomology sheaves are locally constant on all strata.

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2. Motivation of Theorem 1.4

In the introduction we applied our main Theorem 1.4 only to the case of a partial flag variety G/P, although it applies to a homogenous variety of product type $X = Ab \times G/P$, with Ab an abelian variety. We now explain how it is partially motivated by generic vanishing theorems for (semi)abelian varieties. Here a semiabelian variety N fits into an algebraic group extension $0 \to (\mathbb{C}^*)^n \to N \to Ab \to 0$ of an abelian variety by a complex torus. The generic vanishing theorem was first developed in the coherent setting by Green and Lazarsfeld [GL87], and it was adapted to perverse sheaves and D-modules in [GL96, Sch15, LMW19] and other works. One particular generic vanishing theorem for perverse sheaves is as follows.

Theorem 2.1 [LMW19]. *Let N be a semiabelian variety, and let $\mathcal{P}$ be a perverse sheaf on N. Then for a general rank one local system L on N,*

$$H^i(N, \mathcal{P} \otimes L) = 0 \quad \text{for any } i \neq 0.$$

Corollary 2.2. *Let $\mathcal{P}$ be a perverse sheaf on a semiabelian variety N. Then $\chi(N, \mathcal{P}) \geq 0$.*

Proof. Since taking the tensor product with a rank one local system does not change Euler characteristics,

$$\chi(N, \mathcal{P}) = \chi(N, \mathcal{P} \otimes L) \geq 0,$$

where L is a general rank one local system on N. □

The non-negativity of the Euler characteristic of perverse sheaves on a semiabelian variety was also observed by Franecki and Kapranov using the corresponding effective characteristic cycles and Kashiwara's index theorem [FK00] (see also [AMSS22a, Proposition 8.4] for a more general version for suitable varieties mapping to an abelian variety Ab). In the case of an abelian variety it more generally holds for a constructible function φ with an effective characteristic cycle, that is, a non-negative linear combination of signed Euler obstructions $(-1)^{\dim Z} \cdot Eu_Z$ for $Z \subset Ab$ a closed irreducible subvariety. Moreover the corresponding Euler characteristic result $\chi(\varphi) \geq 0$ is only a shadow of the fact that in this case the signed MacPherson Chern class $c^\vee_M(\varphi)$ is effective [AMSS22a, ST10]. Here $c^\vee_M(\varphi)$ differs from $c_M(\varphi)$ by the sign $(-1)^k$ in homological degree $2k$ (for example, they have the same degree zero part).

Remark 2.3. In our new generic vanishing Theorem 1.4 we do not use twisting by a generic rank one local system. Instead we use in particular the affineness assumptions together with a generic translation on the given ambient complete homogeneous variety, in such a way that the underlying constructible functions agree in the sense of Remark 3.6.

Let us illustrate Corollary 1.5 in the simplest case of an abelian (or partial flag) variety given by an elliptic curve E (or by $\mathbb{P}^1$).

Example 2.4. Let $X = E$ be an elliptic curve (or $X = \mathbb{P}^1$) with $A = X \backslash S$ and $B = X \backslash S'$ for two finite non-empty subsets $S, S' \subset X$, with $|S \cup S'| \geq 2$ (as it would be for a generic translate), so that A and B are smooth and affine of codimension zero. Let the perverse sheaf $\mathcal{P} = L[1]$ on A be given by a shifted local system L of rank $r \geq 0$. Then

$$\chi(A \cap B, \mathcal{P}) = -r \cdot \big(\chi(X) - |S \cup S'|\big) \geq r \cdot (|S \cup S'| - 2) \geq 0,$$

since $\chi(E) = 0$ (and $\chi(\mathbb{P}^1)$=2).

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The affine space analog of the work of Franecki–Kapranov was studied using stratified Morse theory in [ST10, STV05].

THEOREM 2.5 [ST10, STV05]. *Let $\mathcal{P}$ be a perverse sheaf on the affine space $\mathbb{C}^n$. Let H be a general affine hyperplane in $\mathbb{C}^n$, and let $U = \mathbb{C}^n \setminus H$. Then*

$$\chi(U, \mathcal{P}|_U) \geq 0.$$

Applying our main Theorem 1.4 to the case when the ambient complete homogeneous variety is $X = \mathbb{P}^n$ and $A = B_0 = \mathbb{C}^n$ gives a new proof of Theorem 2.5. As we have mentioned in Remark 1.8, our theorem in this case is also a result of Beilinson.

Remark 2.6. Theorem 1.4 also directly implies the following counterpart for X a complete homogeneous variety with respect to an action of a connected algebraic group G'. Let $D_1, D_2 \subset X$ be two ample divisors on X, with $A = X \setminus D_1$ and $B_0 = X \backslash D_2$ the affine open complements. Let $\mathcal{P}$ be a perverse sheaf on A. Then

$$\chi(U, \mathcal{P}|_U) \geq 0,$$

for $U := A \backslash gD_2$ the open complement in A of a translate gD_2 of D_2 by a generic element $g \in G'$.

When the perverse sheaf in Corollary 2.2 and Theorem 2.5 is of the form $\mathbb{C}_Y[\dim Y]$, where Y is a smooth subvariety of an affine torus or an affine space, there are explicit geometric interpretations of the Euler characteristics (see [Huh13, RW17], [STV05, Equation (2)] and [ST10, Theorem 1.2]).[1] For a smooth subvariety Y of $(\mathbb{C}^*)^n$, $(-1)^{\dim Y}\chi(Y)$ is equal to the number of critical points of $\prod_{1\leq i\leq n} z_i^{u_i}|_Y$, where z_i are the coordinates of $(\mathbb{C}^*)^n$ and $u_i \in \mathbb{Z}$ are general. Similarly, for a smooth subvariety $Y \subset \mathbb{C}^n$, $(-1)^{\dim Y}\chi(Y \setminus H)$ is equal to the number of critical points of $l|_Y$, where l is a general affine linear function and $H = \{l = 0\}$. These observations lead us to ask the following question.

Question 2.7. Let A and B_0 be two locally closed and pure dimensional smooth affine subvarieties of a complete homogeneous variety X, so that $(-1)^{\dim A\cap B}\chi(A \cap B) \geq 0$ for $B = gB_0$ a generic translate (by Theorem 1.4). Does $(-1)^{\dim A\cap B}\chi(A \cap B)$ count anything?

The answer to the above question potentially will lead to an answer to the following question, which is a constructible function analog of Corollary 1.5.

Question 2.8. Let A, B_0 be locally closed affine irreducible subvarieties of a complete homogeneous variety X, and assume that B_0 is smooth. Let $B = gB_0$ be a generic translate. Does the inequality

$$(-1)^{\dim A\cap B}\chi(Eu_{A\cap B}) \geq 0$$

always hold, where $Eu_{A\cap B}$ denotes the Euler obstruction function of $A \cap B$?

The result of Theorem 2.5 holds for the signed Euler obstruction $(-1)^{\dim Z} \cdot Eu_Z$ of an irreducible algebraic subset $Z \subset \mathbb{C}^n$, since their proof in [ST10, STV05] is given in terms of stratified Morse theory for constructible functions. However, Theorem 1.4 depends essentially on the base change isomorphism (2) in the context of constructible sheaf complexes, which has no counterpart for constructible functions.

[1] If Y is a closed smooth subvariety of a semiabelian variety, then $(-1)^{\dim Y}\chi(Y)$ counts the number of degenerate point of $\eta|_Y$, where η is a general parallel 1-form on the semiabelian variety ([LMW21]). When Y is singular, then the number of critical points in the smooth locus counts the signed Euler characteristic of the Euler obstruction function Eu_Y.

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3. Proof of Theorem 1.4

3.1 A transversality result

We first go over some basic transversality results in the subsection to prepare for the proof of Proposition 1.7.

Lemma 3.1. *Let Z_1 and Z_2 be two locally closed submanifolds of a complex manifold M. Then Z_1 and Z_2 intersect transversally if and only if $Z_1 \times Z_2$ intersects the diagonal $\Delta \subset M \times M$ transversally.*

Proof. The two submanifolds Z_1 and Z_2 intersect transversally at a point $P \in Z_1 \cap Z_2$ if and only if the tangent spaces $T_P Z_1$ and $T_P Z_2$ intersect transversally at P, which is further equivalent to $T_{(P,P)} Z_1 \times Z_2$ intersecting transversally with $T_{(P,P)}\Delta$ in $T_{(P,P)} M \times M$ (see also [Sch03, p. 257] for a discussion of transversal intersections of product stratifications with respect to suitable regularity conditions like the Whitney condition). $\square$

The following is the well-known transversality theorem of Kleiman.

Theorem 3.2 [Kle74]. *Let X be a homogeneous variety with respect to an action of a connected algebraic group G'. Let $\mathfrak{S} = \bigsqcup_{k \in \Lambda_1} S_k$ and $\mathfrak{T} = \bigsqcup_{k' \in \Lambda_2} T_{k'}$ be two Whitney stratifications of X into locally closed smooth complex algebraic subvarieties. For any $g \in G'$, let $g \cdot \mathfrak{T} = \bigsqcup_{k' \in \Lambda_2} gT_{k'}$ be the translate of the stratification $\mathfrak{T}$ by g. Then any stratum in $\mathfrak{S}$ and any stratum in $g \cdot \mathfrak{T}$ intersect transversally for $g \in G'$ generic.*

Combining the above lemma and theorem, we have the following generic transversality result.

Corollary 3.3. *In the notation of the above lemma, the diagonal $\Delta \subset M \times M$ intersects the stratification $\mathfrak{S} \times g \cdot \mathfrak{T}$ of $M \times M$ transversally for $g \in G'$ generic.*

3.2 The proofs

We recall the following special case of [Sch03, Proposition 6.1.1], formulated here only for (the required case of) a complex algebraic Whitney stratification.

Proposition 3.4 (Base change isomorphism). *Let M be a complex algebraic manifold. Let $\mathfrak{S}'$ be a complex algebraic Whitney stratification of the closed algebraic subset $Z' \subset M$, with $A' \subset Z'$ a closed union of strata. Suppose H is a closed algebraic submanifold of M, which is transversal to all strata, and consider the following cartesian diagram of inclusions.*

$$\begin{tikzcd} Z \cap H \arrow[r, "i"] \arrow[d, "k''"] & Z := Z' \backslash A' \arrow[d, "k'"] \\ Z' \cap H \arrow[r, "i'"] & Z' \end{tikzcd} \tag{3}$$

Let $\mathcal{G}$ be a bounded sheaf complex on $Z = Z' \backslash A'$, which is constructible with respect to the induced algebraic Whitney stratification of Z. Then

$$i'^* k'_* \mathcal{G} \cong k''_* i^* \mathcal{G}. \tag{4}$$

Remark 3.5. The result of [Sch03, Proposition 6.1.1] is proven under a weaker stratification condition than Whitney b-regularity, that is, Whitney b-regularity implies (as mentioned in [Sch03, p. 225]) the d^0-regularity used, for example, in [Sch03, Proposition 6.1.1].

We can now prove Proposition 1.7 and then deduce Theorem 1.4.

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Proof of Proposition 1.7. Recall that we have the following commutative diagram.

$$\begin{tikzcd} A \times_X B = A \cap B \arrow[r, "j_B"] \arrow[d, "j_A"] & A \arrow[d, "i_A"] \\ B \arrow[r, "i_B"] & X \end{tikzcd} \tag{5}$$

We need to prove the isomorphism (of derived functors)

$$i_{B!} j_{A*}(\mathcal{F}|_{A\cap B}) \cong i_{A*} j_{B!}(\mathcal{F}|_{A\cap B}). \tag{6}$$

To prove (6), we consider the following diagram, whose intersection with the diagonal gives rise to diagram (5).

$$\begin{tikzcd} A \times B \arrow[r, "\mathrm{id}_A \times i_B"] \arrow[d, "i_A \times \mathrm{id}_B"] & A \times X \arrow[d, "i_A \times \mathrm{id}_X"] \\ X \times B \arrow[r, "\mathrm{id}_X \times i_B"] & X \times X \end{tikzcd} \tag{7}$$

We denote the diagonal map by $\Delta_X : X \to X \times X$, and denote its various restrictions by $\Delta_{A\cap B} : A \cap B \to A \times B$, $\Delta_A : A \to A \times X$, $\Delta_B : B \to X \times B$. These diagonal maps induce a morphism from diagram (5) to diagram (7).

With the above notation, we are ready to prove Proposition 1.7. By [MS22, Proposition 10.2.9], we have the following isomorphism, both sides of which are isomorphic to $\Delta_X^*(i_{A*}\mathcal{F} \boxtimes i_{B!}(\mathbb{K}_B))$:

$$\Delta_X^*(\mathrm{id}_X \times i_B)_!(i_A \times \mathrm{id}_B)_*(\mathcal{F} \boxtimes \mathbb{K}_B) \cong \Delta_X^*(i_A \times \mathrm{id}_X)_*(\mathrm{id}_A \times i_B)_!(\mathcal{F} \boxtimes \mathbb{K}_B). \tag{8}$$

We can simplify the left-hand side of (8) as

$$\begin{aligned} \Delta_X^*(\mathrm{id}_X \times i_B)_!(i_A \times \mathrm{id}_B)_*(\mathcal{F} \boxtimes \mathbb{K}_B) &\cong i_{B!}\Delta_B^*(i_A \times \mathrm{id}_B)_*(\mathcal{F} \boxtimes \mathbb{K}_B) \\ &\cong i_{B!} j_{A*} \Delta_{A\cap B}^*(\mathcal{F} \boxtimes \mathbb{K}_B) \\ &\cong i_{B!} j_{A*}(\mathcal{F}|_{A\cap B}), \end{aligned}$$

where the first isomorphism follows from the base change isomorphism for the direct image with proper support ([Dim04, Theorem 2.3.26]), the second follows from Proposition 3.4, and the last is obvious. Let us check that the transversality assumptions of Proposition 3.4 hold here. Let $\mathfrak{S}$ be a Whitney stratification of X such that $\mathcal{F}$ is constructible with respect to $\mathfrak{S}$ and A is a union of strata. Similarly, let $\mathfrak{T}$ be a Whitney stratification of X such that B_0 is a union of strata. Now, $\mathfrak{S} \times g \cdot \mathfrak{T}$ is a Whitney stratification of $X \times X$ such that both $A \times B$ and $A \times X$ are unions of strata, and $\mathcal{F} \boxtimes \mathbb{K}_B$ is constructible with respect to $\mathfrak{S} \times g \cdot \mathfrak{T}$. Thus, by Corollary 3.3, the assumptions of Proposition 3.4 hold. Also notice that in Proposition 3.4, the statement requires the corresponding embedding k' to be an open embedding. Nevertheless, it also applies to the case when k' is a locally closed embedding by factoring it as the open inclusion to the closure, which is also a union of strata, and the closed embedding to the ambient space, for which the required isomorphism is just the base change isomorphism for the direct image with proper support.

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Similarly, we can simplify the right-hand side of (8) by

$$\begin{aligned}\Delta_X^*(i_A\times \mathrm{id}_X)_*(\mathrm{id}_A\times i_B)_!(\mathcal{F}\boxtimes\mathbb{K}_B) &\cong i_{A*}\Delta_A^*(\mathrm{id}_A\times i_B)_!(\mathcal{F}\boxtimes\mathbb{K}_B)\\ &\cong i_{A*}j_{B!}\Delta_{A\cap B}^*(\mathcal{F}\boxtimes\mathbb{K}_B)\\ &\cong i_{A*}j_{B!}(\mathcal{F}|_{A\cap B}),\end{aligned}$$

where the first isomorphism follows from Proposition 3.4, the second follows from the base change isomorphism for the direct image with proper support, and the last is obvious. By the above simplifications of both sides of (8), we deduce isomorphism (6). □

Proof of Theorem 1.4. Let $d = \operatorname{codim}_X B$, with B smooth of pure codimension. Since B is a generic translate and therefore transversal, $\mathcal{P}|_{A\cap B}[-d]$ is a perverse sheaf (see, for example, [MS22, Proposition 10.2.27] or [Sch03, Lemma 6.0.4]). Then Proposition 1.7 implies by the compactness of X,

$$\begin{aligned}H_c^i\big(B, j_{A*}(\mathcal{P}|_{A\cap B}[-d])\big) &\cong H^i\big(X, i_{B!}j_{A*}(\mathcal{P}|_{A\cap B}[-d])\big)\\ &\cong H^i\big(X, i_{A*}j_{B!}(\mathcal{P}|_{A\cap B}[-d])\big)\\ &\cong H^i\big(A, j_{B!}(\mathcal{P}|_{A\cap B}[-d])\big).\end{aligned}\tag{9}$$

Since both j_A and j_B are quasi-finite and affine, both $j_{A*}(\mathcal{P}|_{A\cap B}[-d])$ and $j_{B!}(\mathcal{P}|_{A\cap B}[-d])$ are perverse sheaves (see, for example, [MS22, Example 10.2.38, Theorem 10.3.69] and [Sch03, Corollary 6.0.5, Theorem 6.0.4]). Thus, by Artin's vanishing theorem (see, for example, [MS22, Theorem 10.3.59] and [Sch03, Corollary 6.0.4]), we have

$$H_c^i\big(B, j_{A*}(\mathcal{P}|_{A\cap B}[-d])\big) = 0,\quad \text{for any } i<0,$$

and

$$H^i\big(A, j_{B!}(\mathcal{P}|_{A\cap B}[-d])\big) = 0,\quad \text{for any } i>0.$$

Combining with isomorphism (9), we have

$$H_c^i\big(B, j_{A*}(\mathcal{P}|_{A\cap B}[-d])\big) \cong H^i\big(A, j_{B!}(\mathcal{P}|_{A\cap B}[-d])\big) = 0,\quad \text{for any } i\neq 0,$$

which is the desired vanishing after a shift of degree. □

Remark 3.6. Using composition of functors, we have

$$H_c^i(B, j_{A!}(\mathcal{P}|_{A\cap B})) = H_c^i(A\cap B, \mathcal{P}|_{A\cap B})$$

and

$$H^i(A, j_{B*}(\mathcal{P}|_{A\cap B})) = H^i(A\cap B, \mathcal{P}|_{A\cap B}).$$

We do not claim that any of them is concentrated in one degree. Our result implies that

$$\chi_c(B, j_{A!}(\mathcal{P}|_{A\cap B})) = \chi(A\cap B, \mathcal{P}|_{A\cap B}) = \chi(A, j_{B*}(\mathcal{P}|_{A\cap B}))$$

has the same sign as $(-1)^{\operatorname{codim} B}$, by the above discussions.

4. The Euler characteristics

In this section we prove Corollary 1.5 and Conjecture 1.1.

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Proof of Corollary 1.5. By Theorem 1.4, we have

$$(-1)^{\operatorname{codim} B}\chi_c\big(B, j_{A*}(\mathcal{P}|_{A\cap B})\big) \geq 0.$$

By [MS22, Equation (10.3) and Example 10.4.37] and [Sch03, §§ 2.3 and 6.0.6]), we have

$$\chi_c\big(B, j_{A*}(\mathcal{P}|_{A\cap B})\big) = \chi\big(B, j_{A*}(\mathcal{P}|_{A\cap B})\big) = \chi(A\cap B, \mathcal{P}|_{A\cap B}).$$

Thus, the claimed inequality of Corollary 1.5 follows. □

Proof of Conjecture 1.1. Let $A = X^\circ_\lambda \cap gX^\circ_\mu$ and $B_0 = X^\circ_{\nu'}$. By Kleiman's theorem, A is smooth and pure dimensional, so that $\mathcal{P} := \mathbb{K}_A[\dim A]$ is a perverse sheaf. By Corollary 1.5,

$$(-1)^{\operatorname{codim} B}\chi(A\cap B, \mathcal{P}|_{A\cap B}) \geq 0.$$

Since $\mathcal{P} = \mathbb{K}_A[\dim A]$, we have $\mathcal{P}|_{A\cap B} = \mathbb{K}_{A\cap B}[\dim A]$, and hence

$$\begin{aligned}(-1)^{\operatorname{codim} B}\chi(A\cap B, \mathcal{P}|_{A\cap B}) &= (-1)^{\dim A - \operatorname{codim} B}\chi(A\cap B)\\ &= (-1)^{\dim A\cap B}\chi(A\cap B)\\ &= (-1)^d\chi(X^\circ_\lambda \cap gX^\circ_\mu \cap hX^\circ_{\nu'})\\ &= E_{\lambda,\mu,\nu'}.\end{aligned}$$

Therefore, we have $E_{\lambda,\mu,\nu'} \geq 0$. □

Theorem 1.6 can be proved in the same way, as explained before that theorem.

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Conflicts of interest

None.

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Jörg Schürmann jschuerm@uni-muenster.de
Mathematisches Institut, Universität Münster, Einsteinstr. 62, 48149 Münster, Germany

Connor Simpson connorgs@connorgs.net
Institute for Advanced Study, 1 Einstein Dr, Princeton, NJ 08540, USA

Botong Wang wang@math.wisc.edu
Institute for Advanced Study, 1 Einstein Dr, Princeton, NJ 08540

and

Department of Mathematics, University of Wisconsin-Madison, 480 Lincoln Drive, Madison, WI 53706-1388, USA

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