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INTERACTING NONLINEAR REINFORCED STOCHASTIC PROCESSES: SYNCHRONIZATION OR NON-SYNCHRONIZATION

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Abstract

The *rich-get-richer rule* reinforces actions that have been frequently chosen in the past. What happens to the evolution of individuals' inclinations to choose an action when agents interact? Interaction tends to homogenize, while each individual dynamics tends to reinforce its own position. Interacting stochastic systems of reinforced processes have recently been considered in many papers, in which the asymptotic behavior is proven to exhibit almost sure synchronization. In this paper we consider models where, even if interaction among agents is present, absence of synchronization may happen because of the choice of an individual nonlinear reinforcement. We show how these systems can naturally be considered as models for coordination games or technological or opinion dynamics.

Keywords: Interacting agents; interacting stochastic processes; reinforced stochastic process; urn model; nonlinear Pólya urn; generalized Pólya urn; reinforcement learning; stochastic approximation; game theory; technological dynamics; CODA models

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1. Introduction

The stochastic evolution of systems composed of agents which interact with each other has always been of great interest in several scientific fields. For example, the economic and social sciences deal with agents that make decisions under the influence of other agents or of external media. Moreover, preferences and beliefs are partly transmitted by means of various forms of social interaction, and opinions are driven by *social influence*, i.e. the tendency of individuals to conform to the majority as they interact with others (e.g. [10, 11, 22, 64]).

A natural description of such systems is provided by agent-based modeling [77, 16], where they are modeled as a collection of decision-making agents with a set of rules (defined at a

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microscopic level) that include several issues, for instance learning and adaptation, environmental constraints, and so on [16, 23, 42]. The character of interaction and influence among agents (or among groups of agents) is crucial in these models and gives rise to emergent phenomena observed in the systems [12, 13]. Agent-based models abound in a variety of disciplines, including biology, economics, game theory, sociology, and political science (e.g. [24, 26, 41, 52, 51, 50, 59, 48, 78, 46, 72]). Although they are often effective in describing real situations, these models are mainly computational. Indeed, because of the many variables involved, it is usually hard to prove analytic results in a rigorous way. On the other hand, the mathematical literature can be a source of inspiration for improving these models, since theoretical results may shed light on aspects that are difficult to capture with a purely numerical approach. For example, many mathematical results in the context of urn models have been used to design and study agent-based models both analytically and computationally.

From a mathematical point of view, there is a growing interest in systems of *interacting reinforced* stochastic processes of different kinds (e.g. [4, 5, 6, 7, 18, 28, 30, 34, 35, 37, 54, 57, 65, 73, 76, 67, 66]). Our work is part of this strand of the literature. Generally speaking, by reinforcement in a stochastic dynamics we mean any mechanism for which the probability that a given event occurs increases with the number of times the same event occurred in the past. This *reinforcement mechanism*, also known as the *rich get richer rule* or *Matthew effect*, is a key feature governing the dynamics of many biological, economic, and social systems (e.g. [75]). The best-known example of a reinforced stochastic process is the standard Eggenberger–Pólya urn (see [45, 68, 79]), which has been widely studied and generalized (some recent variants can be found in [3, 8, 9, 20, 29, 33, 36, 56, 62]).

In this work we consider a system of $N \geq 2$ interacting stochastic processes $\{I^h = (I_{n,h})_{n\geq 1} : 1 \leq h \leq N\}$ such that each one of them takes values in $\{0, 1\}$ and their evolution is modeled as follows: for any $n \geq 0$, the random variables $\{I_{n+1,h} : 1 \leq h \leq N\}$ are conditionally independent given the past information $\mathcal{F}_n$, and for all $h \in \{1, \ldots, N\}$,

$$P_{n,h} = P\big(I_{n+1,h} = 1 \,|\mathcal{F}_n\big) = \alpha Z_n + \beta q + (1 - \alpha - \beta) f(Z_{n,h}), \tag{1}$$

where $\alpha,\ \beta \in [0, 1)$, $\alpha + \beta \in (0, 1)$, $q \in (0, 1]$, the function f is a strictly increasing $[0, 1]$-valued function belonging to $\mathcal{C}^1([0, 1])$,

$$Z_{n+1,h} = (1 - r_n)\, Z_{n,h} + r_n I_{n+1,h} \qquad \text{with } r_n \sim \frac{1}{n}\,, \tag{2}$$

$$Z_n = \frac{1}{N}\sum_{i=1}^{N} Z_{n,i},$$

and $r_n \sim 1/n$ means $\lim_{n\to\infty} n r_n = 1$. The starting point for the dynamics (2) is a random variable $Z_{0,h}$ with values in $[0, 1]$, and the past information $\mathcal{F}_n$ formally corresponds to the σ-field $\sigma\big(Z_{0,h} : 1 \leq h \leq N\big) \vee \sigma\big(I_{k,h} : 1 \leq k \leq n,\ 1 \leq h \leq N\big) = \sigma\big(Z_{k,h} : 0 \leq k \leq n,\ 1 \leq h \leq N\big)$. Summing up, the system represents a population of N interacting units, whose state at time n is synthesized by the pair of random variables $\big(I_{n,h},\ Z_{n,h}\big)$ driven by Equations (1) and (2). As we will explain in more detail below, the fact that f is assumed to be strictly increasing gives rise to a reinforcement mechanism for the individual dynamics.

As a first possible interpretation, let us assume that we are modeling a system of N agents, who at each time-step n have to choose an action $s \in \{0, 1\}$. Suppose that $s = 1$ represents the 'right' choice, that is, the one that gives the greater payoff, and 0 represents the 'wrong' one. For any fixed h, $1 \leq h \leq N$, the process I^h describes the sequence of actions chosen by agent

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h along the time-steps; that is, $I_{n,h}$ is the indicator function of the event 'the agent h makes the right choice at time n'. The process $Z^h = (Z_{n,h})_{n\geq 0}$, with values in [0, 1], can be interpreted as the 'personal inclination' of the agent h in adopting the right choice along time. Therefore, the above model includes three issues:

- Conditional independence of the agents given the past: given the past information up to time n, the agent h makes a choice at time $n+1$ independently of the other agents' choices at time $n+1$.
- At each time $n+1$, the probability $P_{n,h}$ that the agent h makes the right choice is a convex combination of the average value Z_n of all the current agents' inclinations, an external 'forcing input' q, and a function of her own current inclination $Z_{n,h}$. In the sequel, we will refer to this last factor as the 'personal inclination component' of $P_{n,h}$. The term Z_n provides a *mean-field interaction* among the agents. Note that when $\alpha = 0$ there is no interaction: the agents are subject to the same forcing input q, evolving independently of each other. We exclude the case $\alpha = \beta = 0$ because it corresponds to N independent agents who evolve only according to the personal inclination component. We also exclude the case $\alpha + \beta = 1$ because it means that there is no personal inclination component.
- Since f is strictly increasing, there is a reinforcement mechanism on the personal inclination component: if $I_{n,h} = 1$, then $Z_{n,h} > Z_{n-1,h}$ (provided $Z_{n-1,h} < 1$) and so $f(Z_{n,h}) > f\big(Z_{n-1,h}\big)$. In other words, the fact that the agent h makes the right choice at time n implies a positive increment of her inclination towards the adoption of the right choice in the future. As a consequence, the larger the number of times an agent has made the right choice up to time n, the higher her personal inclination component in (1) towards that choice at time $n+1$. The justification of this mechanism is twofold: first, higher payoffs can be related to better physiological conditions, and so individuals who are better fed and healthier are less likely to make mistakes in future choices; second, if the choice is always related to the same action, agents that earn higher payoffs are not inclined to change their action (see [21] and references therein).
- The forcing input q models the presence of an external force (e.g. a political constraint, or an advertising campaign) that leads agents towards the right choice with probability q.

The model considered here also fits well in a different context, where there is no 'right' choice, but agents have to choose between two brands $s \in \{0, 1\}$ that are related to a loyalty program: the more times they select the same brand, the more loyalty points they gain. This fact motivates the reinforcement mechanism on the personal inclination component, and, as above, the forcing input can be interpreted as the possible presence of an external force that leads agents towards the brand 1 with probability q.

Other interpretations can be given in the context of coordination games or opinion dynamics; these will be described in more detail in Sections 1.1 and 1.2 below, where we will focus on specific choices of the function f.

The main object of our study is to check whether the system has long-run equilibrium configurations as $n \to +\infty$—that is, whether, for $h = 1, \ldots, N$, the stochastic process $Z^h = (Z_{n,h})_n$ converges almost surely, as n tends to $+\infty$, to some random variable $Z_{\infty,h}$. Second, we want to analyze the limit configurations $[Z_{\infty,1}, \ldots, Z_{\infty,N}]$, characterizing the

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support of their probability distribution. In particular, we are interested in the phenomenon of *synchronization* of the stochastic processes Z^h, which occurs when all the stochastic processes Z^h converge almost surely towards *the same* random variable. Regarding this question, we point out that the above model with f equal to the identity function is essentially included in the models considered in [4, 34], and in this case, almost sure asymptotic synchronization always take place (precisely, almost sure synchronization towards a random variable when $\alpha > 0$ and $\beta = 0$ and towards the constant q when $\beta > 0$). Vice versa, the systems of interacting reinforced processes studied in [4, 34], for $r_n \sim 1/n$, basically correspond to the model introduced in this paper by taking f equal to the identity function and replacing Z_n with a weighted average of the agents' inclinations.

Synchronization phenomena are ubiquitous in nature and have been observed in a wide variety of models based on randomly interacting units (see the literature cited above and the references therein). In those models, synchronization comes as a result of the interaction and can be enhanced if a reinforcement mechanism is present in the dynamics: for example, in [34] it has been shown that, if reinforcement is sufficiently strong, agents coordinate with each other on a time scale smaller than the one needed to reach their common (random) limit, which gives rise to synchronized fluctuations.

Note that the emergence of collective self-organized behaviors in social communities has been frequently described in models based on a statistical physics approach (e.g. [31, 32, 1, 2, 27]) as a result of a large-scale limit. However, we emphasize that synchronization is not a large-scale phenomenon in the models studied in this paper. Indeed, for suitable values of the parameters, it occurs for *any* value of N. In particular, we will prove that for the models under consideration a *phase transition* occurs, depending on the parameter α that tunes the strength of interaction. When α is close enough to 1, synchronization occurs for any N (even in the absence of the external input). On the other hand, if α is below a certain threshold, 'fragmentation' appears in the population, and several limiting configurations, where agents are divided into two separate groups with two different inclinations, are possible. In this last scenario, the strength of interaction, even if too weak to produce synchronization, still continues to affect the dynamics, through the number of possible limiting configurations and the localization of the limit values for the inclinations.

We point out that in the models studied in [37, 34, 35, 76, 67, 4, 7], cases of non-synchronization may occur only in the absence of interaction, that is, when agents are divided into two or more groups and at least two of these groups behave independently—in other words, when $\alpha = 0$ or when the matrix describing the strengths of interaction between the various agents is not irreducible. By contrast, in the present work, using classical tools of stochastic approximation, we show that cases of non-synchronization may also occur when $\alpha > 0$, i.e. when all the agents interact with each other.

As we will see, the synchronization or non-synchronization of the system is related to the properties of the function f. In particular, in order to have a non-zero probability of non-synchronization, a necessary condition is that f is *not linear*.

Finally, it is worthwhile to note that the asymptotic behavior of the stochastic process Z^h is strictly related to that of the stochastic process $\left\{\bar{I}_n^h = \sum_{k=1}^n I_{k,h}/n\right\}$ (see also [5, 6]), that is, according to the previous interpretation, the average number of times the agent h adopts the right choice. Therefore, the synchronization or non-synchronization of the inclinations of the agents corresponds to the synchronization or non-synchronization of the temporal frequencies with which the agents made the right choice.

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TABLE 1. Payoff matrix. Left: original payoffs. Right: standardized payoffs.

	+1	–1		+1	–1
+1	$a_{+1,+1}$	$a_{+1,-1}$	+1	1	0
–1	$a_{-1,+1}$	$a_{-1,-1}$	–1	A	B

1.1. Interacting systems of coordination games

In this section we illustrate possible interpretations of our model in the context of game theory. Following the approach of [49, 60], each interacting unit h represents a time-evolving 'economy', i.e. a community of agents that grows in time and plays a cooperative game. The whole system describes a population of N communities subject to a *mean-field* interaction and to the influence of an external input. The individual evolution of a given community is defined as follows. At time $n = 0$, there are $N_0 > 0$ agents in the community. Each agent is fully described by a binary pure strategy $s \in S = \{-1, +1\}$. Thus, at any time n, the state of a given community can be characterized by the current share $X_n \in [0, 1]$ of agents playing the strategy $+1$. The system evolves as follows. Given some initial share X_0, at any $n > 0$ a new agent enters the community, observes the current strategy share, and irreversibly chooses a strategy on the basis of expected payoffs. More precisely, call $\pi_n(s)$ the expected payoff associated to the strategy $s \in S$ at time n, and set $\pi_n = \{\pi_n(s) : s \in S\}$. We assume that the probability, say P_n, that the agent n chooses $s = +1$ is a function of π_n. Moreover, we assume that the expected payoffs $\pi_n(s)$ are related to a symmetric 2×2 coordination game; that is, we assume that the agent entering at time n plays a symmetric 2×2 coordination game against all the agents present, according to a standard stage-game payoff matrix as in Table 1.

We assume $a_{+1,+1} > a_{-1,+1}$ and $a_{+1,-1} < a_{-1,-1}$, because the game is a coordination game. We also assume $a_{+1,+1} \geq a_{-1,-1}$ and $a_{+1,-1} \leq a_{-1,+1}$. In what follows, we shall focus on the standardized version of the payoff matrix, obtained from the former (without loss of generality) by letting $A = (a_{-1,+1} - a_{+1,-1})/(a_{+1,+1} - a_{+1,-1}) \in [0, 1)$ and $B = (a_{-1,-1} - a_{+1,-1})/(a_{+1,+1} - a_{+1,-1}) \in (0, 1]$.

For the agent entering at time $n + 1$, the expected payoff associated to any given choice $s \in S$ is given by

$$\pi_n(s) = \begin{cases} X_n & \text{if } s = +1, \\ AX_n + B(1 - X_n) & \text{if } s = -1. \end{cases}$$

Therefore, since P_n is a function of π_n, we get that P_n is a function of X_n, i.e. $P_n = f(X_n)$. The dynamics of the process $(X_n)_n$ is easily given by

$$X_{n+1} = \left(1 - \frac{1}{N_0 + n + 1}\right) X_n + \frac{1}{N_0 + n + 1} I_{n+1}, \tag{3}$$

where I_{n+1} is the indicator function of the event 'the agent entering the community at time $n + 1$ chooses the strategy $+1$', and so $P\big(I_{n+1} = 1 \mid X_k,\ k \leq n\big) = P_n = f(X_n)$. Different individual decision rules give different functions f. Two examples are the following:

- Linear probability (LP):

$$P_n = \frac{\pi_n(+1)}{\pi_n(+1) + \pi_n(-1)},$$

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which gives

$$f(x) = \begin{cases} x & \text{if} \quad A = 0 \text{ and } B = 1, \\ \frac{x}{\theta(x+x^*)} & \text{if} \quad \theta = (1 + A - B), \end{cases} \tag{4}$$

with $x^* = B/\theta = B/(1 + A - B)$ and so $\theta x^* \in (0,\, 1]$ and $\theta x^* \geq 1 - \theta$.

- Logit probability (LogP):

$$P_n = \frac{\exp(K\pi_n(+1))}{\exp(K\pi_n(+1)) + \exp(K\pi_n(-1))},$$

with $K > 0$, which gives

$$f(x) = \frac{1}{1 + \exp(-\theta(x - x^*))}, \tag{5}$$

with $\theta = K(1 - A + B) > 0$ and $x^* = KB/\theta = B/(1 - A + B) \in (0,\, 1)$.

Under the above individual decision rules, long-run equilibria for one community have been studied in [49]:

- With the LP rule, and if the game is not a pure-coordination game (that is, $A = 0$ and $B = 1$), the long-run behavior of the system becomes predictable (see definition in Section 2 below): the share of agents playing $+1$ in the limit converges almost surely to the constant $z_\infty = (1 - B)/(1 + A - B)$. Note that when $B = 1$ we have $z_\infty = 0$, and when $A = 0$ we have $z_\infty = 1$. In all the other cases (that is, $B < 1$ or $A > 0$), coexistence of strategies characterizes the equilibrium configuration, and we have $z_\infty > 1/2$, $z_\infty = 1/2$, or $z_\infty < 1/2$ if and only if $A + B < 1$, $A + B = 1$, or $A + B > 1$, respectively.

  With the LP rule, if the game is a pure-coordination game, then $(X_n)_n$ follows the dynamics of the standard Pólya urn model, and so it converges almost surely to a random variable with beta distribution.

- With LogP, it has been proven that the long-run behavior of the community with $x^* = 1/2$ is predictable if $KB = \theta x^* = \theta/2 \leq 2$: the share of agents playing $+1$ in the limit converges almost surely to $1/2$, which means coexistence of the two strategies in the proportions $1 : 1$. Moreover, when $x^* \neq 1/2$, some numerical analyses have been performed pointing out the coexistence of strategies in the limiting configuration and the fact that the dynamics is again predictable when $KB = \theta x^*$ is small.

We are interested in analyzing the long-run behavior of a system of $N \geq 2$ interacting communities of the above type. More precisely, for each $h \in \{1, \ldots, N\}$, let $Z_{n,h}$ be the share of agents playing the strategy $+1$ in the community h. According to (3), the dynamics for each $Z_{n,h}$ is of the form

$$Z_{n+1,h} = (1 - r_n)\, Z_{n,h} + r_n I_{n+1,h} \qquad \text{with } r_n = \frac{1}{N_0 + n + 1}, \tag{6}$$

where $I_{n+1,h}$ is the indicator function of the event 'the agent entering community h at time $n + 1$ chooses the strategy $+1$'. We assume that $\{I_{n+1,h} : h = 1, \ldots, N\}$ are conditionally independent given the past information $\mathcal{F}_n$ with

$$P_{n,h} = P\big(I_{n+1,h} = 1\,|\mathcal{F}_n\big) = \alpha Z_n + \beta q + (1 - \alpha - \beta) f(Z_{n,h}), \tag{7}$$

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where $Z_n = \sum_{i=1}^{N} Z_{n,i}/N$, $q \in (0, 1]$, and $\alpha,\ \beta \in [0, 1)$ with $\alpha + \beta \in (0, 1)$. This corresponds to assuming that the agent entering a given community at time $n+1$ will choose (independently of the choices of the agents entering the other communities at time $n+1$) the strategy $+1$ with a probability $P_{n,h}$ which is a convex combination of three factors: the present share Z_n of players playing $+1$ in the entire system formed by the N communities, an external forcing input q, and the expected payoffs related to the present share $Z_{n,h}$ of players playing $+1$ in the specific community where the agent enters.

Game theory literature typically focuses on isolated repeated games. The study of interacting games is recent (see, e.g., [39] and the references therein). For instance, in [39], agents interact with each other over multiple channels, where each channel is a repeated game. In a *multichannel game*, an agent repeatedly plays a finite number of independent games. At each round $n+1$, she simultaneously determines how to act in each game, leveraging also the information related to the other games up to round n. Our model can be also interpreted in this way: indeed, consider an agent playing N games, and suppose that at round $n+1$ she chooses for game h (independently of the choices for the other games) the strategy $+1$ with a probability $P_{n,h}$ which is given by a convex combination of three factors: the mean number Z_n of times she played $+1$ in the N games up to round n, an external forcing input q, and a function of the mean number $Z_{n,h}$ of times she played $+1$ in the specific game h up to round n.

It is worthwhile to notice that, although our focus is on the case $N \geq 2$, we are also going to completely describe the asymptotic behavior of the model when $N = 1$. We point out that, in [49], the case $x^* \neq 1/2$ is studied only by means of simulations, while here we provide analytic results.

1.2. Technological and opinion dynamics

By technological dynamics we mean models which describe the diffusion of some technological assets in a given community. Such diffusion may depend on several factors, such as communication between agents, the influence of external media, and a form of self-reinforcement due to agents' loyalty. On the other hand, opinion dynamics deals with the study of formation and evolution of opinions in a population, which is governed by similar factors; in particular, self-reinforcement can be interpreted in this context as a mechanism for which the agents' personal inclination, after being verbalized through the choice of one out of two (or more) possible actions, is subject to reinforcement in the direction of the expressed choice. Therefore, in what follows, we will refer to the first context with the implicit assumption that everything can be translated into the language of the second one. In the above setting, a unit h of our model may be interpreted either as a single agent, to which is associated an opinion or inclination $Z_{n,h}$ to adopt one of two different assets (or actions), or as a whole community of agents which has an internal evolution, driven by the function f, and interacts with other similar communities, eventually under the influence of certain external media. Below, in order to motivate specific choices of the function f, we describe in detail a model based on this last interpretation, where each unit h is modeled as a generalized Pólya urn. In the context of opinion dynamics, our model belongs to the recently studied class of CODA models (continuous opinions, discrete actions) [69, 70].

The generalized Pólya urn model [14, 15, 58] has been used to model the competitive process among new technologies, which is a fundamental phenomenon in economics [40]. This model corresponds to the stochastic process described by (3). Taking f strictly increasing means that the technologies under consideration show increasing returns to adoption: the more they are adopted, the more is learned about them, and, consequently, the more they are improved and the more attractive they become [15]. The dynamics for a single 'market' of potential adopters is as follows: at each time-step n an agent enters the system and decides to adopt one of two possible technologies $s \in \{0, 1\}$ according to the dynamics (3) with a given urn function f. In this framework, it is quite natural to think of the existence of several markets governed by the above dynamics, and to introduce an interaction term between them (see e.g. [19, 71]). The phenomenon of synchronization may be read as the effect of a form of 'contagion' among such markets. The present work is related to the study of the long-run behavior of a system of $N \geq 2$ interacting markets of potential adopters, and so it is described by Equations (6) and (7). An example of a function f that can be used in this setting is given by

$$f(x) = (1-\theta) + (2\theta - 1)\big(3x^2 - 2x^3\big) \qquad \text{with } \theta \in [0, 1], \tag{8}$$

which belongs to $\mathcal{C}^1$ and is strictly increasing when $\theta \in (1/2, 1]$. The applicative justification behind this function is as follows (see [14, 40]). Suppose we have two competing technologies, say $s \in \{0, 1\}$, and represent the community of adopters who are already using one of the two technologies as an urn containing balls of two different colors, say red for technology 1 and black for the other. The composition of the urn evolves with time according to the following decision-making rule for the agents: at each time-step, the agent extracts, with replacement, a random sample of 3 balls from the urn (this means that the agent asks 3 previous agents which technology they are using). Then the agent selects with probability θ the technology used by the majority of the extracted sample (upon which an additional ball of the corresponding color is put into the urn) and with probability $(1-\theta)$ the technology used by the minority of them (upon which an additional ball of the corresponding color is put into the urn). The parameter θ describes the agents' attitude within a single market (which may be cooperative or competitive). A further development of our model of interacting markets can be to consider different values of θ, making them dependent on the market h. Notice that, rephrasing the above description in the language of opinion dynamics, we get a variant of the celebrated Galam majority-rule model [55], with the introduction of a reinforcement mechanism in the dynamics.

According to this dynamics, denoting by T_n and by R_n respectively the total number of balls and the total number of red balls in the urn at time-step n, we have

$$P_n = P\big(I_{n+1} = 1 | \mathcal{F}_n\big) = \theta p\big(T_n, R_n\big) + (1-\theta)\big(1 - p\big(T_n, R_n\big)\big) = (1-\theta) + (2\theta - 1)p(T_n, R_n)$$
$$\text{with } p\big(T_n, R_n\big) = \sum_{k=2}^{3} \frac{\binom{R_n}{k}\binom{T_n - R_n}{3-k}}{\binom{T_n}{3}} \sim \sum_{k=2}^{3} \binom{3}{k} \left(\frac{R_n}{T_n}\right)^k \left(1 - \frac{R_n}{T_n}\right)^{3-k} \quad \text{for } n \to +\infty.$$

(The above approximation is due to the following property of the gamma function: $\Gamma(\mathrm{n+1}) = n!$ and $\Gamma(n+a) \sim n^a \Gamma(n)$ for $n \to +\infty$.) In other words, for $X_n = R_n/T_n$ being the proportion of red balls in the urn at time-step n (i.e. the present share of agents who have adopted technology 1), we have

$$P_n \sim (1-\theta) + (2\theta - 1) \sum_{k=2}^{3} \binom{3}{k} X_n^k (1 - X_n)^{3-k} = f(X_n) \quad \text{for } n \to +\infty,$$

where f is the function given in (8).

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In the case of a single market (i.e. when $N = 1$), studied in [40], the authors find the threshold $1/2$, below which the limit market is shared by the two technologies in the proportions $1:1$. Although our focus is on the case $N \geq 2$, we are also going to completely describe the asymptotic behavior of the model when $N = 1$. In particular, we will correct the abovementioned threshold. Indeed, we will prove that for $N = 1$, when $1/2 < \theta \leq 5/6$, the configuration $1:1$ is the unique limiting configuration, while, when $5/6 < \theta < 1$, two asymmetric limit configurations are possible. Therefore the threshold is not at $1/2$ but at $5/6$.

The rest of the paper is organized as follows. In Section 2 we provide some general results regarding the asymptotic behavior of the systems under consideration. More precisely, we give sufficient conditions for the almost sure convergence of the stochastic processes $Z^h = (Z_{n,h})$ to some random variable $Z_{\infty,h}$ and for the almost sure asymptotic synchronization of the system. Moreover, we give some results concerning the possible values that the limit random vector $[Z_{\infty,1}, \ldots, Z_{\infty,N}]$ can take. In Section 3 we analyze the systems associated to the functions introduced in Subsections 1.1 and 1.2. Specifically, we prove the almost sure convergence of the system, and we characterize the possible limit configurations. In particular, we show sufficient conditions on the parameters to guarantee the almost sure asymptotic synchronization of the system, and we discuss conditions under which the system is predictable (i.e., it has a unique possible limit configuration). Moreover, in relation to the applicative contexts described in Subsections 1.1 and 1.2, we investigate the possible synchronization towards the value $1/2$, which means that, within each community, the two choices asymptotically coexist in the proportions $1:1$. The paper is enriched by simulations and figures, all collected in Section 4, and by an appendix containing some known results from stochastic approximation theory and some technical linear algebra results.

2. General results

By means of (1) and (2), the recursive equation for $Z_{n,h}$ can be rewritten as

$$Z_{n+1,h} = Z_{n,h} + r_n \left[\alpha Z_n + \beta q + (1-\alpha-\beta) f(Z_{n,h}) - Z_{n,h}\right] + r_n \Delta M_{n+1,h}\,, \tag{9}$$

where $\Delta M_{n+1,h} = I_{n+1,h} - P_{n,h}$ is a martingale difference with respect to $\mathcal{F} = (\mathcal{F}_n)_n$. Moreover, summing over h, we get the following equation for Z_n:

$$Z_{n+1} = Z_n + r_n \left[\alpha Z_n + \beta q + (1-\alpha-\beta)\frac{1}{N}\sum_{h=1}^{N} f(Z_{n,h}) - Z_n\right] + r_n \left(\frac{1}{N}\sum_{h=1}^{N}\Delta M_{n+1,h}\right). \tag{10}$$

Let us set $\mathbf{Z}_n = \left(Z_{n,1}, \ldots, Z_{n,N}\right)^{\top}$, $\Delta\mathbf{M}_{n+1} = \left(\Delta M_{n+1,1}, \ldots, \Delta M_{n+1,N}\right)^{\top}$, and

$$\mathbf{F}(\mathbf{z}) = (F_1(\mathbf{z}), \ldots, F_N(\mathbf{z}))^{\top}$$
$$\text{with } F_h(\mathbf{z}) = \alpha \frac{1}{N}\sum_{i=1}^{N} z_i + \beta q + (1-\alpha-\beta) f(z_h) - z_h \quad \forall \mathbf{z} \in [0,1]^N\,. \tag{11}$$

Using the above notation, we can write (9) in the vectorial form

$$\mathbf{Z}_{n+1} = \mathbf{Z}_n + r_n \mathbf{F}(\mathbf{Z}_n) + r_n \Delta\mathbf{M}_{n+1}\,. \tag{12}$$

We are interested in proving that

$$\mathbf{Z}_n \xrightarrow{a.s.} \mathbf{Z}_\infty\,, \tag{13}$$

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where $\mathbf{Z}_\infty$ is a suitable random variable with values in $[0, 1]^N$, and in characterizing the support of its distribution. In particular, when the limit $\mathbf{Z}_\infty$ is of the form $Z_\infty\mathbf{1}$, where Z_∞ is a suitable random variable taking values in [0, 1] and $\mathbf{1}$ is the vector with all components equal to one, we say that the system *almost surely asymptotically synchronizes* (or, briefly, almost surely synchronizes). Indeed, in this case, all the stochastic processes $Z^h = (Z_{n,h})_n$, $h = 1, \ldots, N$, converge almost surely towards the same random variable Z_∞. Further, when Z_∞ is a unique deterministic point, we say that the system is *predictable*.

Finally, it is worthwhile to note that the almost sure convergence of $\mathbf{Z}_n$ towards a random variable $\mathbf{Z}_\infty$ implies the almost sure convergence of the empirical means $\bar{\mathbf{I}}_n = \frac{1}{n}\sum_{k=1}^n \mathbf{I}_k$ (where $\mathbf{I}_k$ is the random vector with components $I_{k,h}$, for $h = 1, \ldots, N$) towards the same limit.

Let $\mathbf{F}$ and $(\mathbf{Z}_n)_{n\geq 0}$ be defined as in (11) and (12). Moreover, using the symbol $\mathbf{0}$ to denote the vector with all components equal to zero, let $\mathcal{Z}(\mathbf{F}) = \{\mathbf{z} \in [0, 1]^N : \mathbf{F}(\mathbf{z}) = \mathbf{0}\}$ be the zero-set of the function $\mathbf{F}$. Further, let us call *synchronization points* the points of the form $\mathbf{z}_\infty = z_\infty\mathbf{1}$, with $z_\infty \in [0, 1]$. Using the stochastic approximation methodology (see Appendix A), we obtain the following results. The first one concerns the almost sure convergence of the process $(\mathbf{Z}_n)$.

Theorem 1. (Almost sure convergence.)
The following statements hold true:

- *The set $\mathcal{Z}(\mathbf{F})$ contains at least one synchronization point.*
- *If $\mathcal{Z}(\mathbf{F})$ is finite and f is integrable, then we have*

$$\mathbf{Z}_n \xrightarrow{a.s.} \mathbf{Z}_\infty,$$

where $\mathbf{Z}_\infty$ is a suitable random variable with values in $\mathcal{Z}(\mathbf{F})$. Moreover, we also have

$$\bar{\mathbf{I}}_n = \frac{1}{n}\sum_{k=1}^n \mathbf{I}_k \xrightarrow{a.s.} \mathbf{Z}_\infty.$$

Proof. We first show that $\mathcal{Z}(\mathbf{F})$ is non-empty, since it contains at least one synchronization point. Indeed, points in $\mathcal{Z}(\mathbf{F})$ are the solutions in $[0, 1]^N$ of the system of equalities

$$\alpha\frac{1}{N}\sum_{i=1}^N z_i + \beta q + (1-\alpha-\beta)f(z_h) - z_h = 0 \qquad \forall h = 1, \ldots, N. \tag{14}$$

In particular, for the synchronization zero points, that is, for the zero points of the form $\mathbf{z} = z\mathbf{1}$, the above system of equalities reduces to the equation

$$\varphi(z) = f(z) - \frac{(1-\alpha)}{(1-\alpha-\beta)}z + \frac{\beta q}{(1-\alpha-\beta)} = 0. \tag{15}$$

(See Figure 2 and Figure 5 for examples and illustrations.) Therefore, since f takes values in [0, 1], we have $\varphi(0) \geq 0$ and $\varphi(1) \leq 0$. This fact implies that φ always has at least one zero point in [0, 1].

Hence, under the above assumptions, the almost sure convergence of $(\mathbf{Z}_n)$ towards a random variable $\mathbf{Z}_\infty$, taking values in $\mathcal{Z}(F)$, follows immediately from Theorem 11. Indeed, we have

$$\mathbf{F} = -\nabla\mathbf{V} \qquad \text{with}$$

$$\mathbf{V}(\mathbf{z}) = -\frac{\alpha}{2N}\left(\sum_{h=1}^N z_h\right)^2 - \beta q\sum_{h=1}^N z_h - (1-\alpha-\beta)\sum_{h=1}^N \phi(z_h) + \frac{1}{2}\sum_{h=1}^N z_h^2,$$

where ϕ is a primitive function of f.

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Finally, since, for each h, we have

$$E\big[I_{n+1,h}|\mathcal{F}_n\big] = P_{n,h} = \alpha Z_n + \beta q + (1-\alpha-\beta)f(Z_{n,h})$$
$$\xrightarrow{a.s.} \alpha \frac{1}{N}\sum_{i=1}^{N} Z_{\infty,i} + \beta q + (1-\alpha-\beta)f\big(Z_{\infty,h}\big) = Z_{\infty,h},$$

applying Lemma B.1 in [5] (with $c_k = k$, $v_{n,k} = k/n$, and $\eta = 1$), we get that $\frac{1}{n}\sum_{k=1}^{n} I_{k,h} \xrightarrow{a.s.} Z_{\infty,h}$. □

The following theorem provides a sufficient condition for the almost sure synchronization of the system.

Theorem 2. *(Almost sure asymptotic synchronization)*

If $\mathcal{Z}(\mathbf{F})$ contains a finite number of synchronization points, f is integrable, and, for each fixed constant $c \in \left(-\frac{\alpha+\beta}{1-\alpha-\beta}, 0\right)$, the function

$$\widetilde{\varphi}(z) = f(z) - \frac{1}{1-\alpha-\beta} z - c \tag{16}$$

has at most one zero point in $[0, 1]$*, then we have the almost sure asymptotic synchronization of the system, and the limit random variable $\mathbf{Z}_\infty$ is of the form $Z_\infty \mathbf{1}$, where Z_∞ satisfies Equation* (15).

Remark 1. (*Linear case.*) Note that, since c belongs to $\left(-\frac{\alpha+\beta}{1-\alpha-\beta}, 0\right)$, we have $\widetilde{\varphi}(0) > 0$ and $\widetilde{\varphi}(1) < 0$ and so the equation $\widetilde{\varphi} = 0$ always has a solution. The above result requires that this solution is unique. A particular case in which this condition is satisfied is when f is linear. Indeed, if $f : [0, 1] \to [0, 1]$ is linear and strictly increasing, then $f' = \delta \in (0, 1]$ and hence $\delta \neq 1/(1-\alpha-\beta)$. It is worthwhile to observe that, when f is linear, Equation (15) has infinite solutions (and so Theorem 2 does not apply) only when f is the identity function and $\beta = 0$. However, this case is included in [4, 34], where the almost sure asymptotic synchronization is proven also in this case.

Proof of Theorem 2. We first prove that the assumptions of Theorem 2 imply that $\mathcal{Z}(\mathbf{F})$ does not contain 'non-synchronization' points, that is, points that are not synchronization points. To this end, we recall that the set $\mathcal{Z}(\mathbf{F})$ is described by the system of equalities (14). In particular, if $\mathbf{z}^*$ is a solution of the system (14) with $z_h^* \neq z_j^*$ for at least one pair of indices, Equation (14) implies

$$(1-\alpha-\beta)f\big(z_h^*\big) < z_h^* \qquad \forall h = 1, \ldots, N \tag{17}$$

and

$$(1-\alpha-\beta)f\big(z_h^*\big) > z_h^* - \alpha - \beta \qquad \forall h = 1, \ldots, N. \tag{18}$$

Moreover, (14) (written for h and j) also implies

$$\big(z_h^* - z_j^*\big) = (1-\alpha-\beta)\big(f\big(z_h^*\big) - f\big(z_j^*\big)\big) \qquad \forall h, j = 1, \ldots, N\,. \tag{19}$$

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Therefore, for a fixed h, $z_j^* \neq z_h^*$ is a solution of the equation

$$f(z) = \frac{1}{1-\alpha-\beta} z + c$$

where

$$c = f\big(z_h^*\big) - \frac{z_h^*}{1-\alpha-\beta} \in \left(-\frac{\alpha+\beta}{1-\alpha-\beta}, 0\right)$$

(by (17) and (18)). In other words, a necessary condition for the existence of non-synchronization zero points is that there exists $c \in Im\big(f - (1-\alpha-\beta)^{-1} id\big) \cap \big(-\frac{\alpha+\beta}{1-\alpha-\beta}, 0\big)$ such that the function (16) has more than one zero point in [0, 1]. Hence, we can conclude that the assumptions of Theorem 2 imply that $\mathcal{Z}(\mathbf{F})$ contains only synchronization points. Therefore, this set is not empty (see Theorem 1), and by assumption it is finite. Applying Theorem 1, we obtain the almost sure convergence of $\mathbf{Z}_n$ towards a random variable $\mathbf{Z}_\infty$ taking values in the set $\mathcal{Z}(\mathbf{F})$, and so of the form $\mathbf{Z}_\infty = Z_\infty \mathbf{1}$, where Z_∞ satisfies Equation (15). □

Remark 2. (*Existence and characterization of the non-synchronization zero points.*) It is worthwhile to underline that, from the above proof, we obtain that a necessary condition for the existence of non-synchronization zero points of $\mathbf{F}$ is that there exists $c \in Im\big(f - (1-\alpha-\beta)^{-1} id\big) \cap \big(-\frac{\alpha+\beta}{1-\alpha-\beta}, 0\big)$ such that the corresponding function (16) has more than one zero point in [0, 1]. Moreover, if $\mathbf{z}^*$ is a non-synchronization zero point, then, for any fixed component z_h^*, every other component is a solution of $\widetilde{\varphi} = 0$, with $c = f\big(z_h^*\big) - z_h^*/(1-\alpha-\beta) \in Im\big(f - (1-\alpha-\beta)^{-1} id\big) \cap \big(-\frac{\alpha+\beta}{1-\alpha-\beta}, 0\big)$. Conversely, when $\mathbf{z}^*$ is a point with the above property, it is a zero point of $\mathbf{F}$ if and only if (because of (14)) we have

$$\alpha \frac{1}{N} \sum_{i=1}^{N} z_i^* + \beta q + (1-\alpha-\beta) c = 0. \tag{20}$$

We conclude this section by providing a very simple condition that allows us to exclude the linearly unstable zero points (see Appendix A) from the set of possible limit points for the process $(\mathbf{Z}_n)$.

Theorem 3. (Non-convergence towards linearly unstable zero points.) *If* $f \in \mathcal{C}^2$, $f(0) > 0$, *and* $f(1) < 1$, *then, for each* $\mathbf{z} \in \mathcal{Z}(\mathbf{F})$ *which is linearly unstable, we have*

$$P(\mathbf{Z}_n \to \mathbf{z}) = 0.$$

Proof. We can apply Theorem 12 in Appendix A. For a fixed $v \in \mathbb{R}^N$ with $|v| = \sum_{h=1}^N |v_h| = 1$ and $n \in \mathbb{N}$, consider the random variable

$$X_{n+1} = \sum_{h=1}^{N} v_h \Delta M_{n+1,h} = \sum_{h=1}^{N} v_h \big(I_{n+1,h} - P_{n,h}\big),$$

where $P_{n,h} = \alpha Z_n + \beta q + (1-\alpha-\beta) f(Z_{n,h})$. We note that a partition of the sample space is given by the events of the form

$$E_{n+1,H} = \big\{I_{n+1,h} = 1\ \forall h \in H,\ I_{n+1,h} = 0\ \forall h \in H^c\big\},$$

where H is a subset of $\{1, \ldots, N\}$ (the empty set included). Therefore, we can write

$$X_{n+1} = \sum_H \left(\sum_{h \in H} v_h \big(1 - P_{n,h}\big) - \sum_{h \in H^c} v_h P_{n,h} \right) I_{E_{n+1,H}} = \sum_H A_{n,h} I_{E_{n+1,H}},$$

where the first sum is over all possible subsets of $\{1, \ldots, N\}$ (the empty set included). It follows that

$$X^+_{n+1} = \sum_H A^+_{n,H} I_{E_{n+1,H}}$$

and so

$$E\big[X^+_{n+1} \mid \mathcal{F}_n\big] = \sum_H A^+_{n,H} E\big[I_{E_{n+1,H}} \mid \mathcal{F}_n\big] = \sum_H A^+_{n,H} \prod_{h \in H} P_{n,h} \prod_{h \in H^c} \big(1 - P_{n,h}\big)$$

(where we use the convention $\prod = 1$ if H or H^c is empty). Now, by assumption, f has on $[0, 1]$ a minimum value $m = f(0) > 0$ and a maximum value $M = f(1) < 1$. Hence we have

$$0 < (1 - \alpha - \beta)m \leq P_{n,h} \leq \alpha + \beta + (1 - \alpha - \beta)M < 1,$$

and this fact implies $\prod_{h \in H} P_{n,h} \prod_{h \in H^c} \big(1 - P_{n,h}\big) \geq p > 0$ for a suitable constant $p > 0$. Moreover, among the possible H, there is $H_* = \{h \in \{1, \ldots, N\} : v_h \geq 0\}$ (possibly equal to the empty set), and correspondingly we have

$$\begin{aligned} A^+_{n,H_*} = A_{n,H_*} &\geq (1 - \alpha - \beta) \min\{m, 1 - M\} \sum_{h \in H_*} v_h + \sum_{h \in H^c_*} (-v_h) \\ &= (1 - \alpha - \beta) \min\{m, 1 - M\} \sum_{h=1}^{N} |v_h| \\ &= (1 - \alpha - \beta) \min\{m, 1 - M\} > 0. \end{aligned}$$

Thus, the condition (47) of Theorem 12 is satisfied with $C = (1 - \alpha - \beta) \min\{m, 1 - M\}p > 0$, and so $P\big(\mathbf{Z}_\infty = \mathbf{z}\big) = 0$ for all the zero points $\mathbf{z}$ of $\mathbf{F}$ that are linearly unstable. □

3. Specific models

In this section, by means of the above general results, we analyze the asymptotic behavior of the systems related to the functions f_{LP}, f_{LogP}, and f_{Tech} introduced in Section 1 (Subsections 1.1 and 1.2). In Section 4 some associated numerical illustrations will be presented.

3.1. Case $f = f_{LP}$

In this subsection we consider the function

$$f(x) = f_{LP}(x) = \frac{x}{\theta(x + x^*)} \qquad \text{with } \theta > 0,\ \theta x^* \in (0, 1],\ \theta x^* \geq 1 - \theta\,. \tag{21}$$

Note that we exclude the case f_{LP} defined in (4) with $\theta = 0$, because it coincides with the case of a system of interacting Pólya urns with mean-field interaction and with or without a 'forcing input' q, and this model has already been analyzed in [4, 5, 6, 34, 35, 37].

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The following result states that, provided that $\mathbf{Z}_0 \neq \mathbf{0}$ (note that, in applications, we generally have $P(\mathbf{Z}_0 \neq \mathbf{0}) = 1$), we always have the almost sure asymptotic synchronization of the system, and moreover it is predictable.

Theorem 4. *Let* $f = f_{LP}$. *Set*

$$\widehat{P} = \begin{cases} P & \textit{when } \beta > 0, \\ P & \textit{when } \beta = 0 \textit{ and } \theta x^* = 1, \\ P(\cdot|\mathbf{Z}_0 \neq \mathbf{0}) & \textit{when } \beta = 0 \textit{ and } \theta x^* < 1, \end{cases}$$

and

$$z_\infty = \begin{cases} \hat{z} & \textit{when } \beta > 0, \\ \frac{1-\theta x^*}{\theta} & \textit{when } \beta = 0, \end{cases} \tag{22}$$

where $\hat{z} \in (0, 1)$ *depends on the model parameters and is defined as in* (24). *Then, under* $\widehat{P}$, *the system almost surely asymptotically synchronizes and it is predictable: indeed, we have*

$$\mathbf{Z}_n \xrightarrow{a.s.} z_\infty \mathbf{1}$$

and

$$\bar{\mathbf{I}}_n = \frac{1}{n} \sum_{k=1}^{n} \mathbf{I}_k \xrightarrow{a.s.} z_\infty \mathbf{1}.$$

Observe that when $\beta > 0$, the limit point z_∞ belongs to $(0, 1)$. In the first interpretation, this means that in the limit configuration the N agents keep a strictly positive personal inclination for both actions. In the interpretation related to games (see Subsection 1.1), since $Z_{n,h}$ is the present share of agents in community h playing the strategy $+1$, this fact means that in the limiting configuration, both strategies coexist in all N communities. When $\beta = 0$, the limit point z_∞ belongs to the entire interval $[0, 1]$, including the extremes: precisely, it is equal to 0 when $\theta x^* = 1$ and equal to 1 when $\theta x^* = 1 - \theta$. We can say that in these last two cases one of the two strategies is asymptotically predominant with respect to the other. Furthermore, we note that the limit value depends only on θ and x^*, but not on the parameter α that rules the interaction.

Proof of Theorem 4. Let us look for the solutions of the equation $\mathbf{F}(\mathbf{z}) = \mathbf{0}$ in $[0, 1]^N$, that is, of the system (14).

Synchronization zero points. We start by looking for the solutions of (14) of the type $\mathbf{z} = z\mathbf{1}$, that is, for the solution of (15). Taking into account that $f = f_{LP}$, we obtain the second-order equation

$$\widehat{\varphi}(z) = (1-\alpha)\theta z^2 + [(1-\alpha)\theta x^* - \beta\theta q - (1-\alpha-\beta)]z - \beta q \theta x^* = 0. \tag{23}$$

The discriminant associated to this equation is

$$\Delta = [(1-\alpha)\theta x^* - \beta\theta q - (1-\alpha-\beta)]^2 + 4(1-\alpha)\beta\theta^2 q x^*.$$

Hence, when $\beta = 0$ and $\theta x^* < 1$, we have two distinct solutions in $[0, 1]$, namely 0 and $\frac{1-\theta x^*}{\theta}$, while if $\beta = 0$ and $\theta x^* = 1$, we have only one solution $z^* = 0$. When $\beta > 0$, we have $\Delta > 0$ and so there are two distinct solutions of (23). However, we are interested only in solutions belonging to $[0, 1]$. Since $\varphi(0) > 0$ and $\varphi(1) < 0$, there is at least one solution in $(0, 1)$. Moreover, since in Δ we have the term $4(1-\alpha)\beta\theta^2 qx^* > 0$, one of the solutions is obviously strictly negative. Therefore, there is a unique solution in (0, 1), given by

$$\hat{z} = \frac{-[(1-\alpha)\theta x^* - \beta\theta q - (1-\alpha-\beta)] + \sqrt{\Delta}}{2(1-\alpha)\theta}. \tag{24}$$

Summing up, synchronization zero points are of the form $\mathbf{z}^* = z^*\mathbf{1}$ with

$$z^* \begin{cases} \in \left\{0, \frac{1-\theta x^*}{\theta}\right\} & \text{if } \beta = 0, \ \theta x^* < 1, \\ = 0 & \text{if } \beta = 0, \ \theta x^* = 1, \\ = \hat{z} & \text{if } \beta > 0. \end{cases} \tag{25}$$

Non-synchronization zero points. Such zero points do not exist: indeed, writing Equation (16) of Theorem 2 for $f = f_{LP}$, we obtain

$$\theta z^2 + \left[c(1-\alpha-\beta)\theta + \theta x^* - (1-\alpha-\beta)\right]z + c(1-\alpha-\beta)\theta x^* = 0,$$

which, since $c < 0$, admits at most one solution in [0, 1].

Almost sure asymptotic synchronization. We have proven above that the set $\mathcal{Z}(\mathbf{F})$ contains only a finite number of points. Moreover, f admits the primitive function

$$\phi(x) = \frac{1}{\theta}\left[x - x^* \ln(x + x^*)\right] + const.$$

Then, by Theorem 1 and Theorem 2, we can conclude that the system almost surely asymptotically synchronizes:

$$\mathbf{Z}_n \xrightarrow{a.s.} \mathbf{Z}_\infty = Z_\infty \mathbf{1}$$

and

$$\bar{\mathbf{I}}_n = \frac{1}{n}\sum_{k=1}^{n} \mathbf{I}_k \xrightarrow{a.s.} \mathbf{Z}_\infty = Z_\infty \mathbf{1},$$

where Z_∞ can take the values z^* specified above. In particular, when we are in the case $\beta > 0$ or in the case $\beta = 0$ and $\theta x^* = 1$, we have a unique possible value for z^* and so the system is predictable. It remains to prove that, under $\widehat{P} = P(\cdot|\mathbf{Z}_0 \neq \mathbf{0})$, the system is predictable with the unique limit point $\frac{1-\theta x^*}{\theta}\mathbf{1}$. The following step provides the proof of this fact.

Case $\beta = 0$ and $\theta x^* < 1$: predictability under $\widehat{P}$. Let us consider the case $\beta = 0$ and $\theta x^* < 1$, for which we have $\mathcal{Z}(\mathbf{F}) = \left\{\mathbf{0}, \left(\frac{1-\theta x^*}{\theta}\right)\mathbf{1}\right\}$. For $\mathbf{z}^* = z^*\mathbf{1}$, Corollary 4 provides the eigenvalues of $J(\mathbf{F})(\mathbf{z}^*)$, that is,

$$(1-\alpha-\beta)f'\left(z^*\right) - 1 \qquad \text{and} \qquad (1-\alpha-\beta)f'\left(z^*\right) - 1 + \alpha.$$

Now, the eigenvalues for $\mathbf{z}^* = \left(\frac{1-\theta x^*}{\theta}\right)\mathbf{1}$ are $(1-\alpha)\theta x^* - 1 < 0$ and $-(1-\alpha)(1-\theta x^*) < 0$, and so $\mathbf{z}^*$ is strictly stable, while the eigenvalues for $\mathbf{0}$ are $(1-\alpha)(\theta x^*)^{-1} - 1$, which can be positive or negative, and $-(1-\alpha)(1-1/\theta x^*) > 0$, so that $\mathbf{0}$ is linearly unstable. However, we cannot exclude convergence towards $\mathbf{0}$ by means of Theorem 3, because $f(0) = 0$. In any case, we observe that if $\mathbf{Z}_0 \neq \mathbf{0}$, then $\mathbf{Z}_n \neq \mathbf{0}$ for all n. Hence, if we prove for $\mathbf{z}^* = \left(\frac{1-\theta x^*}{\theta}\right)\mathbf{1}$ that

$$\langle \mathbf{F}(\mathbf{z}), \mathbf{z} - \mathbf{z}^* \rangle = \langle \mathbf{F}(\mathbf{z}) - \mathbf{F}(\mathbf{z}^*), \mathbf{z} - \mathbf{z}^* \rangle < 0 \tag{26}$$

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for all $\mathbf{z} = (z_1, \ldots, z_N)^T \in [0, 1]^N \setminus \mathcal{Z}(\mathbf{F})$, then we can conclude by Theorem 10 that, under $\widehat{P} = P(\cdot|\mathbf{Z}_0 \neq \mathbf{0})$, the system is predictable. In order to prove (26), we observe that f' is positive and strictly decreasing on $[0, 1]$ and $f'(z^*) = \theta x^* < 1$ by hypothesis. Then, recalling that $f(z) - f(z^*) < 0$ for $z < z^*$ and using the mean value theorem for $z > z^*$, we obtain that $f(z) - f(z^*) < |z - z^*|$ for all $z \in [0, 1]$, $z \neq z^*$. Then, since $z_h \neq z^*$ for at least one $h \in \{1, \ldots, N\}$, we have

$$\begin{aligned}\langle \mathbf{F}(\mathbf{z}) - \mathbf{F}(\mathbf{z}^*), \mathbf{z} - \mathbf{z}^* \rangle &= \frac{\alpha}{N}\left[\sum_{h=1}^{N}(z_h - z^*)\right]^2 - \sum_{h=1}^{N}(z_h - z^*)^2 \\ &\quad + (1-\alpha)\sum_{h=1}^{N}\big(f(z_h) - f(z^*)\big)(z_h - z^*) \\ &< -(1-\alpha)\sum_{h=1}^{N}(z_h - z^*)^2 + (1-\alpha)\sum_{h=1}^{N}(z_h - z^*)^2 = 0. \qquad (27)\end{aligned}$$

Finally, regarding the almost sure convergence of the empirical means under $\widehat{P}$, we observe that the proof given for Theorem 1 also works with $\widehat{P} = P(\cdot|\mathbf{Z}_0 \neq 0)$, because $\{\mathbf{Z}_0 \neq \mathbf{0}\} \in \mathcal{F}_0$. □

Remark 3. (*Possible asymptotic synchronization towards* $1/2$.) We recall that, in the setting described in Subsection 1.1, the quantity $Z_{n,h}$ is the present share of agents in community h playing the strategy $+1$, and so the almost sure asymptotic synchronization of the system towards the value $1/2$ means that in the limit the two strategies in all the communities coexist in the proportions $1:1$. In this regard, we observe that $(1/2)\mathbf{1}$ is a synchronization zero point for the case $f = f_{LP}$ if and only if we have

$$\begin{aligned} &f_{LP}(1/2) - \frac{1-\alpha}{2(1-\alpha-\beta)} + \frac{\beta q}{1-\alpha-\beta} = 0, \qquad \text{that is,} \\ &\frac{\big(\theta + 2\theta x^*\big)}{2}(1 - \alpha - 2\beta q) = 1 - \alpha - \beta, \end{aligned} \qquad (28)$$

which, in particular, implies $(1-\alpha) > 2\beta[q \vee (1-q)]$ (because $f_{LP}(1/2) \in (0, 1)$). Therefore, only when the condition (28) is satisfied does the system almost surely asymptotically synchronize towards $1/2$. Note that, in the special case when $\beta = 0$ (which includes the case $N = 1$, $\alpha = \beta = 0$, which corresponds to the case studied in [49]), the condition (28) simply becomes $\theta x^* = 1 - \theta/2$.

Applying Theorem 13, we can also provide the rate of convergence of $(\mathbf{Z}_n)$. More precisely, we have the following result.

Remark 4. (*Rate of convergence.*) With the same assumptions and notation as in Theorem 4, we have

$$\Delta M_{n+1,h}\Delta M_{n+1,j} = \big(I_{n+1,h} - P_{n,h}\big)\big(I_{n+1,j} - P_{n,j}\big),$$

where $P_{n,h}$ is defined in (1), and so for $h \neq j$, by conditional independence, we get $E\big[\Delta M_{n+1,h}\Delta M_{n+1,j} \mid \mathcal{F}_n\big] = 0$, while for $h = j$, taking into account that $\mathbf{F}(\mathbf{z}_\infty) = \mathbf{0}$ for $\mathbf{z}_\infty = z_\infty \mathbf{1}$,

$$E\Big[\big(\Delta M_{n+1,h}\big)^2 \mid \mathcal{F}_n\Big] = P_{n,h} - P_{n,h}^2 \xrightarrow{a.s.} z_\infty - z_\infty^2 \qquad \text{with respect to } \widehat{P}.$$

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Moreover, by Corollary 4, the smallest eigenvalue of $-J(\mathbf{F})(z_\infty\mathbf{1})$ is $\lambda=(1-\alpha)-(1-\alpha-\beta)f'(z_\infty)$. Therefore, applying Theorem 13 when $x_\infty\in(0,1)$, we obtain, under $\widehat{P}$, the following:

- If $\lambda>1/2$, then
$$\sqrt{n}(\mathbf{Z}_n-z_\infty\mathbf{1})\xrightarrow{d}\mathcal{N}(\mathbf{0},\Sigma)\,,$$
where, using Remark 16, we obtain $\Sigma=z_\infty(1-z_\infty)\,(-2J(\mathbf{F})(\mathbf{z}_\infty)-Id)^{-1}$.
- If $\lambda=1/2$, then
$$\sqrt{\frac{n}{\ln(n)}}(\mathbf{Z}_n-z_\infty\mathbf{1})\xrightarrow{d}\mathcal{N}(\mathbf{0},\Sigma)\,,$$
where Σ is a suitable matrix of the form $z_\infty(1-z_\infty)\widehat{\Sigma}$.
- If $0<\lambda<1/2$, then
$$n^\lambda(\mathbf{Z}_n-z_\infty\mathbf{1})\xrightarrow{a.s.}V,$$
where V is a suitable finite random variable.

3.2. Case $f=f_{LogP}$

In this subsection, we consider the function

$$f(x)=f_{LogP}(x)=\frac{1}{1+\exp(-\theta(x-x^*))}\qquad\text{with } x^*\in(0,1),\ \theta>0\,. \tag{29}$$

It is a sigmoid function, i.e. its first derivative is a strictly positive function, which is strictly increasing on $[0,x^*)$ and strictly decreasing on $(x^*,1]$, with a maximum given by $f'(x^*)=\theta/4$. Furthermore, we have $f'(x)=f'(2x^*-x)$ for all $x\in[0,1]$.

The following lemma provides a description of the subset of $\mathcal{Z}(\mathbf{F})$ containing all the zero points of $\mathbf{F}$ that are synchronization points (briefly, 'synchronization zero points').

Lemma 1. (Synchronization zero points.) *Let $f=f_{LogP}$. Then, depending on the values of the parameters, $\mathcal{Z}(\mathbf{F})$ contains at least three synchronization zero points. Moreover, at most two of them are stable. In particular, if one of the following conditions is satisfied,* $\mathbf{F}$ *has a unique stable synchronization zero point:*

(U1) $\theta/4\le(1-\alpha)/(1-\alpha-\beta)$, *or*

(U2) $f'(0)\vee f'(1)\ge(1-\alpha)/(1-\alpha-\beta)$, *or*

(U3) $f'(0)\vee f'(1)<(1-\alpha)/(1-\alpha-\beta)<\theta/4$, *and either* $f\big(\widehat{x}_1\big)>(1-\alpha)\widehat{x}_1/(1-\alpha-\beta)-\beta q/(1-\alpha-\beta)$ *or* $f\big(\widehat{x}_2\big)<(1-\alpha)\widehat{x}_2/(1-\alpha-\beta)-\beta q/(1-\alpha-\beta)$, *where* $\widehat{x}_1\in(0,x^*)$ *and* $\widehat{x}_2=2x^*-\widehat{x}_1\in(x^*,1)$ *are the solutions of* $f'=(1-\alpha)/(1-\alpha-\beta)$.

Otherwise, $\mathbf{F}$ *has two stable synchronization zero points belonging to* $\big(0,\widehat{x}_1\big]\cup\big[\widehat{x}_2,1\big)$ *(more precisely, one in each of these two intervals).*

Proof. We recall (see Theorem 1) that there exists at least one synchronization zero point of $\mathbf{F}$, and that points of this type are of the form $\mathbf{z}=z\mathbf{1}$ with $\varphi(z)=0$, where

$$\varphi(z)=f(z)-(1-\alpha)/(1-\alpha-\beta)z+\beta q/(1-\alpha-\beta).$$

Note that $\varphi(z)$ is of the form $f(z) - \delta z + cost$, with $\delta = (1-\alpha)/(1-\alpha-\beta)$ and a suitable constant $cost$ such that $\varphi(0) > 0$ and $\varphi(1) < 0$ (note that $f(0) > 0$ and $f(1) < 1$). Hence, we have $\varphi' = f' - \delta$ and $\varphi'' = f''$. Therefore, recalling that f is a sigmoid function with $\max_{[0,1]} f' = f'(x^*) = \theta/4$, and using the symmetry of f', we get that the equation $\varphi'(x) = 0$, i.e. $f'(x) = \delta$, has at most two solutions on [0, 1], and we have the following cases:

(1) $\theta/4 \leq \delta$;

(2) $f'(0) \vee f'(1) \geq \delta$ (and so $\theta/4 > \delta$);

(3) $f'(0) \vee f'(1) < \delta < \theta/4$, and, letting $\widehat{x}_1 \in (0, x^*)$ and $\widehat{x}_2 = 2x^* - \widehat{x}_1 \in (x^*, 1)$ be the solutions of $\varphi' = 0$, we have either $\varphi(\widehat{x}_1) > 0$ or $\varphi(\widehat{x}_2) < 0$;

(4) $f'(0) \vee f'(1) < \delta < \theta/4$, and, letting $\widehat{x}_1$, $\widehat{x}_2$ be as in (3), we have either $\varphi(\widehat{x}_1) = 0$ or $\varphi(\widehat{x}_2) = 0$;

(5) $f'(0) \vee f'(1) < \delta < \theta/4$, and, letting $\widehat{x}_1$, $\widehat{x}_2$ be as in (3), we have $\varphi(\widehat{x}_1) < 0$ and $\varphi(\widehat{x}_2) > 0$.

(Note that Cases (1), (2) and (3) coincide, respectively, with conditions (U1), (U2) and (U3) in the statement.) In Case (1), φ' is strictly negative on $[0, 1] \setminus \{x^*\}$, that is, φ is strictly decreasing on [0, 1], and, since $\varphi(0) > 0$ and $\varphi(1) < 0$, this fact implies that $\varphi = 0$ has a unique solution in (0, 1). In Case (2), observe that, since $\varphi(0) > 0$ and $\varphi(1) < 0$, it holds that $\varphi'(0) \wedge \varphi'(1) < 0$ (otherwise φ would be increasing on [0, 1], yielding a contradiction). Now, if $\varphi'(0) \geq 0$, then $\varphi'(x) > 0$ for all $x \in (0, x^*]$, i.e. φ is strictly increasing on $(0, x^*]$; this fact implies that $\varphi(x) > 0$ for all $x \in [0, x^*]$. Consequently, φ has at most one zero point $z^* \in (x^*, 1)$, because $\varphi(1) < 0$ and φ' is strictly decreasing on $(x^*, 1]$. Analogously, if $\varphi'(1) \geq 0$, then φ has at most one zero point $z^* \in (0, x^*)$. In Case (3), $\widehat{x}_1 < x^*$ and $\widehat{x}_2 = 2x^* - \widehat{x}_1 > x^*$ are respectively the points of a local minimum and a local maximum of φ in (0, 1). Now, if $\varphi(\widehat{x}_1) > 0$, then φ has a unique zero $z^* \in (\widehat{x}_2, 1)$. Analogously, if $\varphi(\widehat{x}_2) < 0$, then φ has a unique zero $z^* \in (0, \widehat{x}_1)$. In Case (4), the function φ has two zero points: more precisely, if $\widehat{x}_1$ is a zero point of φ, then $\varphi(\widehat{x}_2) > 0$ and the other zero point of φ belongs to $(\widehat{x}_2, 1)$; if $\widehat{x}_2$ is a zero point of φ, then $\varphi(\widehat{x}_1) < 0$ and the other zero point of φ belongs to $(0, \widehat{x}_1)$. Finally, in Case (5), φ has three zero points: one in $(0, \widehat{x}_1)$, one in $(\widehat{x}_1, \widehat{x}_2)$, and the last in $(\widehat{x}_2, 1)$.

Regarding the stability of the synchronization zero points of $\mathbf{F}$, we observe that when $\mathbf{z} = z\mathbf{1}$, where z is a zero point of φ, by Corollary 4, the eigenvalues of $J(\mathbf{F})(\mathbf{z})$ are given by

$$(1-\alpha-\beta)f'(z) - 1 \qquad \text{and} \qquad (1-\alpha-\beta)f'(z) - 1 + \alpha\,,$$

that is,

$$(1-\alpha-\beta)\varphi'(z) - \alpha \qquad \text{and} \qquad (1-\alpha-\beta)\varphi'(z).$$

Therefore, in Cases (1), (2), and (3), the unique synchronization zero point is stable, because the corresponding eigenvalues are both negative. In Case (4), recalling that φ' is strictly negative on $(0, \widehat{x}_1)$ and on $(\widehat{x}_1, 1)$ and strictly positive on $(\widehat{x}_1, \widehat{x}_2)$, both synchronization zero points are stable. For the same reason, in Case (5), the synchronization zero point strictly smaller than $\widehat{x}_1$ and the one strictly bigger than $\widehat{x}_2$ are stable, while the one in $(\widehat{x}_1, \widehat{x}_2)$ is linearly unstable. □

Remark 5. Note that if $x^*\mathbf{1}$ is a synchronization zero point of $\mathbf{F}$, that is, $(1-\alpha)(1-2x^*) - \beta(1-2q) = 0$, then (U2) is not possible, because, as shown in the above proof, in that case the unique zero point of φ is necessarily different from x^*. Moreover, Cases (U3) and (U4)

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are also not possible. Indeed, $f-(1-\alpha)id/(1-\alpha-\beta)$ is strictly increasing on $\big(\widehat{x}_1, \widehat{x}_2\big)$ and so we have $f\big(\widehat{x}_1\big)-(1-\alpha)\widehat{x}_1/(1-\alpha-\beta) < f(x^*)-(1-\alpha)x^*/(1-\alpha-\beta) = -\beta q/(1-\alpha-\beta) < f\big(\widehat{x}_2\big)-(1-\alpha)\widehat{x}_2/(1-\alpha-\beta)$. Therefore, when $x^*\mathbf{1}$ is a synchronization zero point of $\mathbf{F}$, it is stable if and only if (U1) is satisfied. Otherwise, there are three synchronization zero points: x^* (linearly unstable) and two stable, say $\mathbf{z}_1^* = z_1^*\mathbf{1}$ and $\mathbf{z}_2^* = z_2^*\mathbf{1}$, with $0 < z_1^* < \widehat{x}_1 < x^* < \widehat{x}_2 < z_2^* = 2x^* - z_1^* < 1$.

As an immediate consequence of Lemma 1, we get that, if the system almost surely asymptotically synchronizes and one of the conditions (U1), (U2), and (U3) holds true, then it is predictable. In the next results (see Theorems 5 and 6) we give sufficient conditions for the almost sure asymptotic synchronization of the system. Moreover, we provide a characterization of the possible limit points that are not synchronization points (see Theorem 6).

Theorem 5. *Let $f = f_{LogP}$. Assume that one of the following conditions holds:*

- *(S1)* $\theta/4 \le 1/(1-\alpha-\beta)$, *or*
- *(S2)* $f'(0)\vee f'(1) \ge 1/(1-\alpha-\beta)$ *(and so* $\theta/4 > 1/(1-\alpha-\beta)$*), or*
- *(S3)* $f'(0)\vee f'(1) < 1/(1-\alpha-\beta) < \theta/4$, *and, letting* $x_1^* \in (0, x^*)$ *and* $x_2^* = 2x^* - x_1^* \in (x^*, 1)$ *be the solutions of* $f' = 1/(1-\alpha-\beta)$*, we have either* $f\big(x_1^*\big) \ge x_1^*/(1-\alpha-\beta)$ *or* $f\big(x_2^*\big) \le x_2^*/(1-\alpha-\beta) - (\alpha+\beta)/(1-\alpha-\beta)$.

Then we have the almost sure asymptotic synchronization of the system, i.e.

$$\mathbf{Z}_n \xrightarrow{a.s.} \mathbf{Z}_\infty$$

and

$$\bar{\mathbf{I}}_n = \frac{1}{n}\sum_{k=1}^{n} \mathbf{I}_k \xrightarrow{a.s.} \mathbf{Z}_\infty,$$

where $\mathbf{Z}_\infty$ is a random variable of the form $\mathbf{Z}_\infty = Z_\infty\mathbf{1}$. Moreover, the random variable Z_∞ can take at most two different values, both belonging to $(0, 1)$. *(Specifically, $\mathbf{Z}_\infty$ takes values in the set of stable zero points of $\mathbf{F}$, which is contained in* $(0, 1)^N$ *and consists of at most two different points.)*

Proof. We want to apply Theorem 1, Theorem 2, and Theorem 3. Observe first that $f = f_{LogP}$ admits the primitive function

$$\phi(x) = x + \frac{1}{\theta}\ln\big(1 + e^{-\theta(x-x^*)}\big) + const$$

and, by Lemma 1, the set of synchronization zero points of $\mathbf{F}$ is finite. Now, consider the function $\widetilde{\varphi}$ defined in Theorem 2. Observe that this function has the same form of φ: indeed, we have $\widetilde{\varphi}(z) = f(z) - \delta z + cost$, with $\delta = 1/(1-\alpha-\beta)$ and $cost = -c \in \big(0, \frac{\alpha+\beta}{1-\alpha-\beta}\big)$, and so $\widetilde{\varphi}(0) > 0$ and $\widetilde{\varphi}(1) < 0$. Therefore, arguing exactly as in the proof of Lemma 1, with $\widetilde{\varphi}$ in place of φ and $\delta = 1/(1-\alpha-\beta)$, we obtain that each of the above conditions (S1), (S2), and (S3) implies that, for all $c \in \big(-\frac{\alpha+\beta}{1-\alpha-\beta}, 0\big)$, the function $\widetilde{\varphi}$ has exactly one zero point in [0, 1]. Indeed, (S1) and (S2) correspond to the conditions (1) and (2) in the proof of Lemma 1, while the condition (S3) implies, for all $c \in \big(-\frac{\alpha+\beta}{1-\alpha-\beta}, 0\big)$, that $\widetilde{\varphi}$ satisfies the condition (3) in the proof of Lemma 1. Applying Theorem 1 and Theorem 2 we obtain the almost sure asymptotic synchronization of the system, that is, $\mathbf{Z}_n \xrightarrow{a.s.} \mathbf{Z}_\infty = Z_\infty\mathbf{1}$, where Z_∞ takes values in the set of zero

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points of φ. Moreover, recalling that $f(0) > 0$ and $f(1) < 1$, we can also apply Theorem 3 and conclude that the support of the limit random variable $\mathbf{Z}_\infty$ consists of only the zero points of φ that give rise to a stable synchronization zero point of $\mathbf{F}$. By Lemma 1, such points belong to (0, 1), and there are at most two of them. □

The next theorem deals with the case not covered by Theorem 5. In particular, analyzing the stability of eventual 'non-synchronization zero points' of $\mathbf{F}$, we provide another condition under which we have the almost sure asymptotic synchronization of the system (see the condition (S4) below). Moreover, we characterize the possible 'non-synchronization limiting configurations' for the system.

Theorem 6. *Let $f = f_{LogP}$ and suppose*

$$f'(0) \vee f'(1) < \frac{1}{(1-\alpha-\beta)} < \theta/4, \quad f(x_1^*) < \frac{x_1^*}{(1-\alpha-\beta)}, \quad f(x_2^*) > \frac{x_2^* - (\alpha+\beta)}{(1-\alpha-\beta)}, \tag{30}$$

where $x_1^* \in (0, x^*)$ and $x_2^* = 2x^* - x_1^* \in (x^*, 1)$ are the solutions of $f' = 1/(1-\alpha-\beta)$. Moreover, assume that $\mathcal{Z}(\mathbf{F})$ is finite. Then

$$\mathbf{Z}_n \xrightarrow{a.s.} \mathbf{Z}_\infty$$

and

$$\bar{\mathbf{I}}_n = \frac{1}{n}\sum_{k=1}^{n} \mathbf{I}_k \xrightarrow{a.s.} \mathbf{Z}_\infty,$$

where $\mathbf{Z}_\infty$ takes values in the set $\mathcal{SZ}(\mathbf{F})$ of the stable zero points of $\mathbf{F}$, which is contained in $(0, 1)^N$. This set always contains at most two synchronization zero points. Moreover, if $(1-\alpha-\beta)\left[f'(0)+f'(1)\right] < 1 + (1-\alpha)$, any $\mathbf{z}_\infty \in \mathcal{SZ}(\mathbf{F})$ which is not a synchronization point has, up to permutations, the form $\mathbf{z}_\infty = (z_{\infty,1}, \ldots, z_{\infty,N})^T$ with

$$z_{\infty,h} = \begin{cases} \tilde{z}_{\infty,1} \in \left(0, x_1^*\right) & \text{for } h = 1, \ldots, N_1, \\ \tilde{z}_{\infty,2} \in \left(x_2^*, 1\right) & \text{for } h = N_1+1, \ldots, N, \end{cases} \tag{31}$$

and $N_1 \in \{1, \ldots, N-1\}$. On the other hand, if

$$(1-\alpha-\beta)\left[f'(0)+f'(1)\right] \geq 1 + (1-\alpha), \tag{32}$$

then $\mathcal{SZ}(\mathbf{F})$ contains only synchronization points (and so we have the almost sure asymptotic synchronization of the system).

Summing up, taking $f = f_{LogP}$, if we are in one of the previous cases (S1), (S2), and (S3), or if we are in the case, say (S4), when $\mathcal{Z}(\mathbf{F})$ is finite and (30) and (32) are satisfied, then we have the almost sure asymptotic synchronization of the system. Otherwise, the system almost surely converges and asymptotic synchronization is still possible, but the asymptotic non-synchronization of the system cannot be excluded. More precisely, it is possible to observe the system splitting into two groups of components that converge towards two different values. We also point out that the above results state that the random limit $\mathbf{Z}_\infty$ always belongs to $(0, 1)^N$. In the first interpretation, this fact means that in the limiting configuration the N agents always keep a strictly positive personal inclination for both actions. In the interpretation involving games (see Subsection 1.1), since $Z_{n,h}$ is the present share of agents in community h

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who adopted the strategy $+1$, this fact means that in the limiting configuration, both strategies coexist in all N communities. Moreover, regarding the possible 'non-synchronization limiting configurations' we have that, independently from the value of N, we always have at most two groups of components that approach two different values in the limit. We never have a more complicated asymptotic fragmentation of the whole system. Furthermore, we are able to localize the two limit values: one is strictly smaller than $x_1^* < x^*$ and the other strictly bigger than $x_2^* > x^*$, where the points x_i^* depend only on x^*, θ, and $(1-\alpha-\beta)$, which is the 'weight' of the personal inclination component of $P_{n,h}$ in (1). In addition, Remark 6 (after the proof) may provide some information on the sizes of the two groups. In Section 4 some numerical illustrations are presented. In Figure 4 the chosen set of parameters is such that there is only one (stable) synchronization zero point. In Figure 6 ($\beta \neq 0$) and Figure 9 ($\beta = 0$) there are two stable synchronization zero points, and there are stable non-synchronization zero points.

Proof of Theorem 6. Almost sure convergence follows from the fact that f is integrable (see the proof of Theorem 5 above) and $\mathcal{Z}(\mathbf{F})$ is finite, so that we can apply Theorem 1. Moreover, since $f(0) > 0$ and $f(1) < 1$, by Theorem 3 we get that the random variable $\mathbf{Z}_\infty$ takes values in the set of stable points of $\mathcal{Z}(\mathbf{F})$. By Lemma 1, this set always contains one or two synchronization points. Let us now investigate the existence of stable non-synchronization zero points of $\mathbf{F}$. According to Remark 2, a necessary condition for the existence of a solution $\mathbf{z}^* = \left(z_1^*, \ldots, z_N^*\right)^T$ of (14) with $z_h^* \neq z_j^*$ for at least one pair of indices h, j is that there exists $c \in Im\big(f - (1-\alpha-\beta)^{-1} id\big) \cap \left(-\frac{\alpha+\beta}{1-\alpha-\beta}, 0\right)$ such that the corresponding function $\widetilde{\varphi}$ defined in (16) has more than one zero point in [0, 1]. In relation to this, we observe that the assumptions (30) and the fact that $f - (1-\alpha-\beta)^{-1} id$ is continuous and strictly increasing on $\left(x_1^*, x_2^*\right)$ imply that $Im\big(f - (1-\alpha-\beta)^{-1} id\big) \cap \left(-\frac{\alpha+\beta}{1-\alpha-\beta}, 0\right)$ coincides with

$$I = \left(f\left(x_1^*\right) - \frac{x_1^*}{(1-\alpha-\beta)} \vee -\frac{(\alpha+\beta)}{1-\alpha-\beta},\ f\left(x_2^*\right) - \frac{x_2^*}{(1-\alpha-\beta)} \wedge 0 \right),$$

and it is not empty. Moreover, for each c belonging to this set, the corresponding function $\widetilde{\varphi}$ has the same form of φ: indeed, we have $\widetilde{\varphi}(z) = f(z) - \delta z + cost$, with $\delta = 1/(1-\alpha-\beta)$ and $cost = -c$, such that $\widetilde{\varphi}(0) > 0$ and $\widetilde{\varphi}(1) < 0$. Therefore, arguing exactly as in the proof of Lemma 1, with $\widetilde{\varphi}$ in place of φ and $\delta = 1/(1-\alpha-\beta)$, we obtain that the assumptions (30) imply that the equation $\widetilde{\varphi} = 0$ has two or three distinct solutions in (0, 1) (see Cases (4) and (5) in the proof of Lemma 1). More precisely, the equation $\widetilde{\varphi}' = 0$, that is, $f' = \delta$, has exactly two solutions $x_1^*, x_2^* \in (0, 1)$ $\left(\text{with } x_1^* < x^* < x_2^* = 2x^* - x_1^*\right)$, which are respectively the points of a local minimum and a local maximum of $\widetilde{\varphi}$ in (0, 1); moreover, since $c \in I$, we have $\widetilde{\varphi}\left(x_1^*\right) \le 0$ and $\widetilde{\varphi}\left(x_2^*\right) \ge 0$. Therefore, $\widetilde{\varphi}$ has two zero points (Case (4)), one in $\left\{x_1^*, x_2^*\right\}$ and the other in $\left(0, x_1^*\right) \cup \left(x_2^*, 1\right)$, or it has three zero points, one in $\left(x_1^*, x_2^*\right)$ and the other two in $\left(0, x_1^*\right) \cup \left(x_2^*, 1\right)$. Hence, by Remark 2, if $\mathbf{z}^*$ is a non-synchronization zero point of $\mathbf{F}$, then, for a fixed component z_h^*, every other component is a solution of $\widetilde{\varphi} = 0$ with $c = f\left(z_h^*\right) - (1-\alpha-\beta)^{-1} z_h^* \in I$, and so its components belong to (0, 1) and are, up to permutations, of the following form:

$$z_h^* = \begin{cases} \tilde{z}_1 = \zeta_1(\tilde{z}_2) & \text{for } h = 1, \ldots, N_1, \\ \tilde{z}_2 & \text{for } h = N_1 + 1, \ldots, N_1 + N_2, \\ \tilde{z}_3 = \zeta_3(\tilde{z}_2) & \text{for } h = N_1 + N_2 + 1, \ldots, N_1 + N_2 + N_3 = N, \end{cases} \tag{33}$$

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where $N_i \in \{0, \ldots, N-1\}$, $\tilde{z}_1 \le \tilde{z}_2 \le \tilde{z}_3$, $\tilde{z}_2 \in \left[x_1^*, x_2^*\right]$,

$$\zeta_1(\tilde{z}_2) \begin{cases} = \tilde{z}_2 = x_1^* & \text{if } \tilde{z}_2 = x_1^*, \\ < x_1^* & \text{if } \tilde{z}_2 \in \left(x_1^*, x_2^*\right], \end{cases} \qquad \zeta_3(\tilde{z}_2) \begin{cases} = \tilde{z}_2 = x_2^* & \text{if } \tilde{z}_2 = x_2^*, \\ > x_2^* = 2x^* - x_1^* & \text{if } \tilde{z}_2 \in \left[x_1^*, x_2^*\right), \end{cases}$$

and (see (20) in Remark 2)

$$\alpha \frac{1}{N} \left(N_1 \zeta_1(\tilde{z}_2) + N_2 \tilde{z}_2 + N_3 \zeta_3(\tilde{z}_2)\right) + \beta q + (1-\alpha-\beta)c = 0\,. \tag{34}$$

Finally, let us study the stability of such a point. Note that, since $\tilde{z}_2 \in \left[x_1^*, x_2^*\right]$, we have $\widetilde{\varphi}'(\tilde{z}_2) = f'(\tilde{z}_2) - 1/(1-\alpha-\beta) \ge 0$. Moreover, for $\tilde{z}_2 \in \left(x_1^*, x_2^*\right)$, we have $\widetilde{\varphi}'(\tilde{z}_i) = f'(\tilde{z}_i) - 1/(1-\alpha-\beta) < 0$ for $i = 1, 3$, while if $\tilde{z}_2 = x_1^* = \tilde{z}_1$ $\left(\text{respectively, } \tilde{z}_2 = x_2^* = \tilde{z}_3\right)$, we have necessarily $\widetilde{\varphi}'(\tilde{z}_3) = f'(\tilde{z}_3) - 1/(1-\alpha-\beta) < 0$ (respectively, $\widetilde{\varphi}'(\tilde{z}_1) = f'(\tilde{z}_1) - 1/(1-\alpha-\beta) < 0$). Therefore, if $N_2 \neq 0$, we have $f'\left(z_h^*\right) > \left(1 - \frac{\alpha}{N}\right)/(1-\alpha-\beta)$ for all $h \in \{N_1 + 1, \ldots, N_1 + N_2\}$. Now, for $\mathbf{w} = (w_1, \ldots, w_N)^T \in [0,1]^N$, consider

$$\begin{aligned} \left\langle \mathbf{F}(\mathbf{w}), \mathbf{w} - \mathbf{z}^* \right\rangle &= \left\langle \mathbf{F}(\mathbf{w}) - \mathbf{F}(\mathbf{z}^*), \mathbf{w} - \mathbf{z}^* \right\rangle \\ &= \frac{\alpha}{N} \left[\sum_{h=1}^N \left(w_h - z_h^*\right) \right]^2 + (1-\alpha-\beta) \sum_{h=1}^N \left(f(w_h) - f\left(z_h^*\right)\right) \left(w_h - z_h^*\right) \\ &\quad - \sum_{h=1}^N \left(w_h - z_h^*\right)^2. \end{aligned} \tag{35}$$

If we choose an index k such that $z_k^* = \tilde{z}_2$ and we take $w_h = z_h^*$ for all $h \neq k$ and $w_k = \tilde{z}_2 + \epsilon$, with $\epsilon \neq 0$, the above scalar product (35) can be written as

$$-\left(1 - \frac{\alpha}{N}\right)\epsilon^2 + (1-\alpha-\beta) f'(\xi)\epsilon^2\,,$$

with ξ a suitable point in the interval with extremes $\tilde{z}_2$ and $\tilde{z}_2 + \epsilon$. Hence, if we take $\epsilon \neq 0$ sufficiently small so that $f'(\xi) > \left(1 - \frac{\alpha}{N}\right)/(1-\alpha-\beta)$, the above quantity is strictly positive. This fact implies that $\mathbf{z}^*$ is linearly unstable (see Appendix A). A similar argument shows that if $N_2 = 0$, $\tilde{z}_1 = x_1^*$ or $N_2 = 0$, $\tilde{z}_2 = x_2^*$, then the point $\mathbf{z}^*$ is linearly unstable.

Now, let us consider a zero point $\mathbf{z}^*$ of the form (33) with $N_2 = 0$. If $f'(\tilde{z}_1) = f'(\tilde{z}_3)$ we can apply Corollary 4 and conclude that $\mathbf{z}^*$ is stable if and only if $(1-\alpha-\beta) f'(\tilde{z}_1) \le 1 - \alpha$. Since $2(1-\alpha) \le 1 + (1-\alpha)$, this last condition implies

$$(1-\alpha-\beta)[f'(\tilde{z}_1) + f'(\tilde{z}_3)] < 1 + (1-\alpha). \tag{36}$$

If $f'(\tilde{z}_1) \neq f'(\tilde{z}_3)$, Corollary 5 provides conditions for the stability of $\mathbf{z}^*$, and by Remark 18, a necessary condition for the stability of $\mathbf{z}^*$ is given by (53), that is,

$$\alpha \frac{N_i}{N} < 1 - (1-\alpha-\beta) f'(\tilde{z}_i) \qquad \forall i = 1, 3.$$

Since $N_3 = N - N_1$, we find

$$-(1-\alpha) + (1-\alpha-\beta) f'(\tilde{z}_3) < \alpha \frac{N_1}{N} < 1 - (1-\alpha-\beta) f'(\tilde{z}_1). \tag{37}$$

Note that the above inequalities imply the condition (36) again. Moreover, since f' is strictly increasing on $[0, x^*)$ and strictly decreasing on $(x^*, 1]$ and $\tilde{z}_1 < x^* < \tilde{z}_3$, the condition (36) necessarily implies

$$(1-\alpha-\beta)\left[f'(0)+f'(1)\right] < 1+(1-\alpha). \tag{38}$$

Summing up, under the assumptions of the theorem under consideration, if the condition (38) is not satisfied (that is, if (32) is satisfied), then we have the almost sure asymptotic synchronization of the system. Otherwise, if (38) is satisfied, then both synchronization and non-synchronization zero points of $\mathbf{F}$ are possible values for $\mathbf{Z}_\infty$. In particular, the possible non-synchronization values are those of the form (31) (note that $\tilde{z}_{\infty,1} = \tilde{z}_1$ and $\tilde{z}_{\infty,2} = \tilde{z}_3$). □

Remark 6. (*Restrictions on the possible values for* N_1.) Suppose we are under the same assumptions as in Theorem 6. It could be useful to observe that, as seen in the proof of Theorem 6, the relation (37) might provide a restriction on the possible values for N_1. Note that the two bounds depend on the values $\tilde{z}_i$, $i = 1, 3$. However, recalling that f′ is strictly increasing on $[0, x^*)$ and strictly decreasing on $(x^*, 1]$ and $\tilde{z}_1 < x^* < \tilde{z}_3$, from (37) we obtain

$$-(1-\alpha)+(1-\alpha-\beta)f'(1) < \alpha\frac{N_1}{N} < 1-(1-\alpha-\beta)f'(0),$$

which might provide two bounds not depending on the values of the component of the limit point.

In the following remark we discuss the possible asymptotic synchronization of the system towards the value $1/2$.

Remark 7. (*Possible asymptotic synchronization towards* $1/2$.) As previously mentioned, in the setting of Subsection 1.1, the random variable $Z_{n,h}$ represents the present share of agents in community h who have adopted the strategy $+1$, and so almost sure asymptotic synchronization towards the value $1/2$ means that in the limit the two strategies in all the communities coexist in the proportions $1:1$. With $f = f_{LogP}$, the point $1/2\mathbf{1}$ is a synchronization zero point if and only if we have

$$f(1/2)+\frac{2\beta q-(1-\alpha)}{2(1-\alpha-\beta)} = 0, \tag{39}$$

which, since f takes values in $(0, 1)$, implies $(1-\alpha) > 2\beta[q \vee (1-q)]$. Moreover, by Lemma 1, if $(1/2)\mathbf{1}$ is a zero point of $\mathbf{F}$, then it is stable (and so a possible limit point for the system) if and only if one of the conditions (U1) and (U2) is satisfied or when $1/2$ belongs to $\left(0, \widehat{x}_1\right] \cup \left[\widehat{x}_2, 1\right)$ (note that (U3) is included in this last condition).

In the next two remarks, we discuss some of the conditions introduced in the above results, providing simple conditions on x^*, α, and β sufficient to guarantee or to exclude them.

Remark 8. (*Regarding the conditions (S2), (S4), and (U2).*) We show that if α, β, and x^* satisfy a particular condition (see (41) below), the above cases (S2) and (S4) are not possible. Indeed, taking $f = f_{LogP}$, we have $|x - x^*| f'(x) = g(\theta|x - x^*|)$, where

$$g(x) := \frac{x\exp(x)}{(1+\exp(x))^2} = \frac{x\exp(-x)}{(1+\exp(-x))^2}. \tag{40}$$

Therefore, observing that $\max_{[0,+\infty[} g < 1/4$ (see Figure 1), we get that the condition

$$\min\{x^*, (1-x^*)\} \geq \frac{(1-\alpha-\beta)}{4(1-\alpha)} \tag{41}$$

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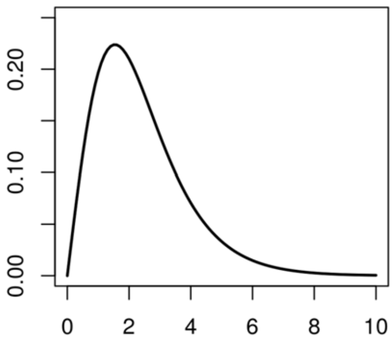

FIGURE 1. Graph of the function g.

implies $f'(0) \vee f'(1) < (1-\alpha)/(1-\alpha-\beta)$. Hence, if (41) holds true, then (S2) and (32) (and so (S4)) are not possible. Furthermore, under (41), Case (U2) of Lemma 1 is also not possible.

Note that the above condition (41) is satisfied when $x^* = 1/2$.

Remark 9. (*Regarding the conditions* (S3) *and* (30).) Take $f = f_{LogP}$ and consider the case $f'(0) \vee f'(1) < \frac{1}{(1-\alpha-\beta)} < \theta/4$. Let $x_1^* < x^* < x_2^*$ be such that $f'\left(x_i^*\right) = 1/(1-\alpha-\beta)$. If x^* belongs to the interval

$$\left(\frac{1}{2} - \frac{(\alpha+\beta)}{2}, \ \frac{1}{2} + \frac{(\alpha+\beta)}{2}\right)$$

(for instance, this is the case when $x^* = 1/2$), then we necessarily have $f\left(x_1^*\right) < x_1^*/(1-\alpha-\beta)$ and $f\left(x_2^*\right) > x_2^*/(1-\alpha-\beta) - (\alpha+\beta)/(1-\alpha-\beta)$. Indeed, if x^* belongs to the above interval, then $f(x^*) - x^*/(1-\alpha-\beta) = 1/2 - x^*/(1-\alpha-\beta)$ belongs to $(-(\alpha+\beta)/(1-\alpha-\beta), \ 0)$, and so, since the function $f - (1-\alpha-\beta)^{-1} id$ is strictly increasing on $\left(x_1^*, x_2^*\right)$, we get the two desired inequalities for $f\left(x_i^*\right) - x_i^*/(1-\alpha-\beta)$, $i = 1, \ 2$. As a consequence, Case (S3) is not possible.

As a consequence of the above results and remarks, we obtain the following corollary, which deals with the special case where $x^* = 1/2$ and either $\beta = 0$ or $q = 1/2$. See Figure 9.

Corollary 1. *(Special case: $x^* = 1/2$ and either $\beta = 0$ or $q = 1/2$) Take $f = f_{LogP}$ with $x^* = 1/2$, and suppose that one of the conditions $\beta = 0$ or $q = 1/2$ is satisfied. Assume $\mathcal{Z}(\mathbf{F})$ is finite. Then, using the same notation as in Lemma 1 and Theorem 6, only the following cases are possible:*

(a) We have $\theta/4 \le (1-\alpha)/(1-\alpha-\beta)$. If this is the case, the system almost surely asymptotically synchronizes, and it is predictable; the unique limit point is $x^* = 1/2$.

(b) We have $(1-\alpha)/(1-\alpha-\beta) < \theta/4 \le 1/(1-\alpha-\beta)$. If this is the case, the system almost surely synchronizes, but there are two possible limit points, $\mathbf{z}_i^* = z_i^* \mathbf{1}$, $i = 1, \ 2$, with $0 < z_1^* < \widehat{x}_1 < 1/2 < \widehat{x}_2 < z_2^* = 1 - z_1^* < 1$.

(c) We have $\theta/4 > 1/(1-\alpha-\beta)$. If this the case, the system almost surely converges to a random variable $\mathbf{Z}_\infty$ taking values in the set $\mathcal{SZ}(\mathbf{F})$ of the stable zero points of $\mathbf{F}$, which is contained in $(0, 1)^N$. This set always contains two stable synchronization zero points, $\mathbf{z}_i^* = z_i^* \mathbf{1}$, $i = 1, \ 2$, with $0 < z_1^* < \widehat{x}_1 < 1/2 < \widehat{x}_2 < z_2^* = 1 - z_1^* < 1$, and any $\mathbf{z}_\infty \in \mathcal{SZ}(\mathbf{F})$

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which is not a synchronization point has the form (31)*, up to permutations. In particular, when*

$$x_1^* \leq x^* - \frac{1-\alpha-\beta}{4(1-\alpha)} = \frac{1}{2} - \frac{1-\alpha-\beta}{4(1-\alpha)}, \tag{42}$$

the points of the form (31) *with* $0 < \tilde{z}_{\infty,1} < x_1^* < 1/2$*,* $1/2 < x_2^* < \tilde{z}_{\infty,2} = 1 - \tilde{z}_{\infty,1} < 1$*,* $N_1 = N/2$*, and* $(1-\alpha-\beta)f(\tilde{z}_{\infty,1}) - \tilde{z}_{\infty,1} = -(\alpha+\beta)/2$ *are stable non-synchronization zero points of* $\mathbf{F}$*.*

(Note that for $\beta = 0$, the above condition (42) becomes simply $x_1^* \leq 1/4$.)

Proof. By Remark 5, if $x^* = 1/2$ and one of the conditions $\beta = 0$ or $q = 1/2$ is satisfied, then $(1/2)\mathbf{1}$ is a synchronization zero point of $\mathbf{F}$. Moreover, it is stable if and only if (U1) is satisfied, and if this is the case, then the system almost surely asymptotically synchronizes, and it is predictable. If (U1) is not satisfied, we have two stable synchronization zero points, whose components are symmetric with respect to $x^* = 1/2$; that is, $\mathbf{z}_i^* = z_i^* \mathbf{1}$, $i = 1,\ 2$, with $0 < z_1^* < \widehat{x}_1 < 1/2 < \widehat{x}_2 < z_2^* = 1 - z_1^* < 1$. Moreover, by Remarks 8 and 9, Cases (S2), (S3), and (S4) are not possible (because $x^* = 1/2$ implies $f'(0) \vee f'(1) < (1-\alpha)/(1-\alpha-\beta)$, $f\big(x_1^*\big) < x_1^*/(1-\alpha-\beta)$, and $f\big(x_2^*\big) > x_2^*/(1-\alpha-\beta) - (\alpha+\beta)/(1-\alpha-\beta)$). Summing up, we can have only the following: Case (S1), in which we have the almost sure asymptotic synchronization of the system and, when (U1) is satisfied, also its predictability (see Cases (a) and (b) in the statement); or the case when (30) is satisfied, but not (32), and so convergence towards non-synchronization zero points of the form (31) may be possible (see case (c) in the statement). Moreover, we observe that when we are in this last case, taking $c = f(x^*) - x^*/(1-\alpha-\beta) = -(\alpha+\beta)/[2(1-\alpha-\beta)]$, the zero points of the corresponding function $\widetilde{\varphi}$ are symmetric with respect to $x^* = 1/2$ (that is, $\widetilde{\varphi}(z) = 0 \Leftrightarrow \widetilde{\varphi}(1-z)$). It follows that points of the form (33) with $0 < \tilde{z}_1 < x_1^* < 1/2$, $\tilde{z}_2 = x^* = 1/2$, $1/2 < x_2^* < \tilde{z}_3 = 1 - \tilde{z}_1 < 1$ are zero points of $\mathbf{F}$ if and only if $f(\tilde{z}_1) - z_1/(1-\alpha-\beta) = c$ and (34) is satisfied, that is,

$$\alpha \frac{1}{N}\big(N_1\tilde{z}_1 + N_2/2 + N_3\tilde{z}_3\big) + \beta q + (1-\alpha-\beta)c = 0\,. \tag{43}$$

In particular, if we take $N_1 = N_3$ and use the symmetry between $\tilde{z}_1$ and $\tilde{z}_3$, we obtain that (43) is satisfied if and only if $\beta = 0$ or $q = 1/2$. Therefore, when $x^* = 1/2$ and one of the conditions $\beta = 0$ or $q = 1/2$ is satisfied, $\mathbf{F}$ has non-synchronization zero points of the form (33) with $0 < \tilde{z}_1 < x_1^* < 1/2$, $\tilde{z}_2 = x^* = 1/2$, $1/2 < x_2^* < \tilde{z}_3 = 1 - \tilde{z}_1$, $N_1 = N_3$, and $f(\tilde{z}_1) - z_1/(1-\alpha-\beta) = c$. Now, such points are stable if and only if $N_2 = 0$ (and so $N_1 = N_3 = N/2$) and $(1-\alpha-\beta)f'(\tilde{z}_1) \leq 1-\alpha$ (note that we are in the case $f'(\tilde{z}_1) = f'(\tilde{z}_3)$). Since f' is strictly increasing on $[0, x^*)$, this last condition is satisfied when $(1-\alpha-\beta)f'\big(x_1^*\big) \leq 1-\alpha$. Finally, using the fact that, for $f = f_{LogP}$, we have $f'(x) = g(\theta|x-x^*|)/|x-x^*|$, where g is the function defined in (40) and such that $\max_{[0,+\infty[}\, g < 1/4$, to guarantee the stability of the non-synchronization zero points considered, it is enough to require (42). □

We conclude this section with two remarks: one regarding the case $N = 1$ and the other concerning the rate of convergence.

Remark 10. *(Case $N = 1$.)* This remark is devoted to the case $N = 1$ and its relationship with the results obtained in [49].

The above proofs (with the due simplifications) also work in the case $N = 1$ and $\alpha = \beta = 0$, which corresponds to the case studied in [49]. Indeed, in this case, we have to consider only Theorem 1, Lemma 1, and Remark 5 (with $N = 1$ and $\alpha = \beta = 0$). As a consequence, when

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$x^* = 1/2$, if $\theta \le 4$, then the system is predictable and the unique limiting configuration is given by $z_\infty = 1/2$; otherwise it almost surely converges, but it is not predictable, and the two possible limiting configurations belong to $(0, \widehat{x}_1] \cup [\widehat{x}_2, 1) \subset (0, 1) \setminus \{1/2\}$ and are symmetric with respect to 1/2. This is the same as the result obtained in [49]. For the case $x^* \neq 1/2$, in [49] there are only numerical analyses, while here we have proven a precise result: if one of the conditions (U1), (U2), or (U3) is satisfied, then the system is predictable and the limiting configuration belongs to $(0, 1) \setminus \{x^*, 1/2\}$ (more precisely, it is strictly smaller than 1/2 when $x^* > 1/2$ and strictly greater than 1/2 when $x^* < 1/2$, because $\varphi(1/2) = f(1/2) - 1/2 = f(1/2) - f(x^*)$ and f is strictly increasing); otherwise it almost surely converges, but the system is not predictable, and the two possible limit configurations belong to $(0, 1) \setminus \{x^*, 1/2\}$ (more precisely, one belongs to $(0, \widehat{x}_1] \setminus \{1/2\} \subset (0, x^*) \setminus \{1/2\}$ and one to $[\widehat{x}_2, 1) \setminus \{1/2\} \subset (x^*, 1) \setminus \{1/2\}$, and so, as before, taking into account that $\varphi(1/2) = f(1/2) - f(x^*)$, one is strictly smaller than 1/2 and the other strictly greater than 1/2).

Remark 11. (*Rate of convergence.*) When the system is predictable with $\mathbf{z}_\infty = z_\infty \mathbf{1}$ as the unique possible limit value for $\mathbf{Z}_\infty$, applying the same arguments used in Remark 4, we can obtain a central limit theorem where the rate of convergence is driven by $\lambda = (1 - \alpha) - (1 - \alpha - \beta) f'(z_\infty)$ (see Theorem 13 and Remark 16).

When the system almost surely converges to $\mathbf{Z}_\infty$, but it is not predictable, applying Remark 17, we get $1/\sqrt{n}$ as the rate of convergence, for any $\mathbf{z}_\infty$ with $P(\mathbf{Z}_n \to \mathbf{z}_\infty) > 0$ and $\lambda(\mathbf{z}_\infty) > 0$. In particular, with some computations similar to those in Remark 4, we have that the matrix $\Gamma = \Gamma(\mathbf{z}_\infty)$ is a diagonal matrix with diagonal elements equal to $z_{\infty,h}\big(1 - z_{\infty,h}\big)$ with $z_{\infty,h} = z_\infty$ (synchronization point) or $z_{\infty,h} \in \{\tilde{z}_{\infty,1}, \tilde{z}_{\infty,2}\}$ (non-synchronization point). Finally, note that when $\mathbf{z}_\infty$ is a possible non-synchronization limit point with $\tilde{z}_{\infty,2} = 2x^* - \tilde{z}_{\infty,1}$, we have $f'(\tilde{z}_{\infty,1}) = f'(\tilde{z}_{\infty,2})$ and so again $\lambda(\mathbf{z}_\infty) = (1 - \alpha) - (1 - \alpha - \beta) f'(\tilde{z}_{\infty,1})$.

3.3. Case $f = f_{\text{Tech}}$

In this subsection we consider the function f_{Tech} defined in (8) with $\theta \in (1/2, 1)$, that is,

$$f(x) = f_{\text{Tech}}(x) = (1 - \theta) + (2\theta - 1)\big(3x^2 - 2x^3\big) \qquad \text{with } \theta \in (1/2, 1). \tag{44}$$

Similarly to f_{LogP}, the function $f = f_{\text{Tech}}$ is a sigmoid function, i.e. its first derivative is a strictly positive function which is strictly increasing on $[0, 1/2)$ and strictly decreasing on $(1/2, 1]$, with a maximum given by $f'(1/2) = 3\theta - 3/2$. Furthermore, we have $f'(x) = f'(1 - x)$ for all $x \in [0, 1]$. Differently from f_{LogP}, we have $f'(0) = f'(1) = 0$. Therefore, arguing exactly as in the proof of Lemma 1 and Remark 5, but using the fact that $x^* = 1/2$ and $f'(0) = f'(1)$, we obtain the following lemma and remark.

Lemma 2. (Synchronization zero points.) *Let $f = f_{\text{Tech}}$. Then, depending on the values of the parameters, $\mathcal{Z}(\mathbf{F})$ contains at least three synchronization zero points. Moreover, at most two of them are stable. In particular, if one of the following conditions is satisfied,* $\mathbf{F}$ *has a unique stable synchronization zero point:*

(U1) $3\theta - 3/2 \le (1 - \alpha)/(1 - \alpha - \beta)$, *or*

(U2) $(1 - \alpha)/(1 - \alpha - \beta) < 3\theta - 3/2$, *and either* $f\big(\widehat{x}_1\big) > (1 - \alpha)/(1 - \alpha - \beta)\widehat{x}_1 - \beta q/(1 - \alpha - \beta)$ *or* $f\big(\widehat{x}_2\big) < (1 - \alpha)/(1 - \alpha - \beta)\hat{x}_2 - \beta q/(1 - \alpha - \beta)$, *where* $\widehat{x}_1 \in (0, 1/2)$ *and* $\widehat{x}_2 = 1 - \widehat{x}_1 \in (1/2, 1)$ *are the solutions of* $f' = (1 - \alpha)/(1 - \alpha - \beta)$.

Otherwise, $\mathbf{F}$ *has two stable synchronization zero points belonging to* $(0, \widehat{x}_1] \cup [\widehat{x}_2, 1)$ *(more precisely, one in each of these two intervals).*

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Remark 12. Note that if $1/2\mathbf{1}$ is a synchronization zero point of $\mathbf{F}$, that is, $\beta=0$ or $q=1/2$, then (U2) is not possible. Indeed, $f-(1-\alpha)id/(1-\alpha-\beta)$ is strictly increasing on $(\widehat{x}_1, \widehat{x}_2)$ and so we have

$$\begin{aligned} f(\widehat{x}_1)-(1-\alpha)\widehat{x}_1/(1-\alpha-\beta) &< f(1/2)-(1-\alpha)/2(1-\alpha-\beta) \\ &= -\beta q/(1-\alpha-\beta) \\ &< f(\widehat{x}_2)-(1-\alpha)\widehat{x}_2/(1-\alpha-\beta). \end{aligned}$$

Moreover, when $1/2\mathbf{1}$ is a synchronization zero point of $\mathbf{F}$, it is stable if and only if (U1) is satisfied. Otherwise, there are three synchronization zero points: $1/2$ (linearly unstable) and two that are stable, say $\mathbf{z}_1^*=z_1^*\mathbf{1}$ and $\mathbf{z}_2^*=z_2^*\mathbf{1}$, with $0<z_1^*<\widehat{x}_1<1/2<\widehat{x}_2<z_2^*=1-z_1^*<1$.

As an immediate consequence of Lemma 2, we get that if the system almost surely asymptotically synchronizes and one of the conditions (U1) or (U2) holds true, then it is predictable.

Now, we observe that $f=f_{\rm Tech}$ admits the primitive function

$$\phi(x)=(1-\theta)x+(2\theta-1)\left(1-\frac{x}{2}\right)x^3+const.$$

Moreover, for $\theta\in(1/2, 1)$, we have $f(0)=1-\theta>0$ and $f(1)=\theta<1$. Therefore, arguing exactly as in the proof of Theorem 5, but with some simplifications due to the fact that $x^*=1/2$ (see Remark 9) and $f'(0)=f'(1)=0$, we obtain the following result.

See Figure 10, Figure 11, and Figure 12 in Section 4 for associated simulations and illustrations.

Theorem 7. *Let $f=f_{\rm Tech}$. If $3\theta-3/2\le 1/(1-\alpha-\beta)$, then we have the almost sure asymptotic synchronization of the system, i.e.*

$$\mathbf{Z}_n \xrightarrow{a.s.} \mathbf{Z}_\infty$$

and

$$\bar{\mathbf{I}}_n=\frac{1}{n}\sum_{k=1}^{n}\mathbf{I}_k \xrightarrow{a.s.} \mathbf{Z}_\infty,$$

where $\mathbf{Z}_\infty$ is a random variable of the form $\mathbf{Z}_\infty=Z_\infty\mathbf{1}$. Moreover, the random variable Z_∞ can take at most two different values, both belonging to $(0, 1)$. *(Specifically, $\mathbf{Z}_\infty$ takes values in the set of stable zero points of* $\mathbf{F}$, *which is contained in* $(0, 1)^N$ *and consists of at most two different points.)*

The next theorem deals with the case not covered by Theorem 7. In particular, we characterize the possible non-synchronization limiting configurations for the system. The proof is exactly the same as that given for Theorem 6, but taking into account that $x^*=1/2$ and $f'(0)=f'(1)=0$ and, above all, that $\mathcal{Z}(\mathbf{F})$ is finite (see Lemma 4 in the appendix).

Theorem 8. *Let $f=f_{\rm Tech}$. If $1/(1-\alpha-\beta)<3\theta-3/2$, then*

$$\mathbf{Z}_n \xrightarrow{a.s.} \mathbf{Z}_\infty$$

and

$$\bar{\mathbf{I}}_n=\frac{1}{n}\sum_{k=1}^{n}\mathbf{I}_k \xrightarrow{a.s.} \mathbf{Z}_\infty,$$

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where $\mathbf{Z}_\infty$ *takes values in the set* $\mathcal{SZ}(\mathbf{F})$ *of the stable zero points of* $\mathbf{F}$*, which is contained in* $(0, 1)^N$*. This set always contains at most two synchronization zero points. Moreover, any* $\mathbf{z}_\infty \in \mathcal{SZ}(\mathbf{F})$ *which is not a synchronization point has, up to permutations, the form* $\mathbf{z}_\infty = (z_{\infty,1}, \ldots, z_{\infty,N})^T$ *with*

$$z_{\infty,h} = \begin{cases} \tilde{z}_{\infty,1} \in \left(0, x_1^*\right) & \text{for } h = 1, \ldots, N_1, \\ \tilde{z}_{\infty,2} \in \left(x_2^*, 1\right) & \text{for } h = N_1 + 1, \ldots, N, \end{cases} \tag{45}$$

where $x_1^* \in (0, 1/2)$ *and* $x_2^* = 1 - x_1^* \in (1/2, 1)$ *are the solutions of* $f' = 1/(1 - \alpha - \beta)$*, and* $N_1 \in \{1, \ldots, N - 1\}$.

Since f_{Tech} is a polynomial of degree 3, Lemma 4 implies that the set $\mathcal{Z}(\mathbf{F})$ is finite. Summing up, taking $f = f_{\text{Tech}}$, if $3\theta - 3/2 \le 1/(1 - \alpha - \beta)$, then we have almost sure asymptotic synchronization of the system. Otherwise, the system almost surely converges and asymptotic synchronization is still possible, but the asymptotic non-synchronization of the system cannot be excluded. More precisely, it is possible to observe the system splitting into two groups of components that converge towards two different values. We also point out that the above results state that the random limit $\mathbf{Z}_\infty$ always belongs to $(0, 1)^N$. In the first interpretation, this fact means that in the limit configuration the N agents always keep a strictly positive personal inclination for both actions. In the interpretation involving technological dynamics (see Subsection 1.2), since $Z_{n,h}$ is the present share of agents in market h who have adopted technology 1, this fact means that in the limiting configuration, both technologies coexist in all N markets. Moreover, regarding the possible non-synchronization limiting configurations, we have that, independently of the value of N, we always have at most two groups of components that approach two different values in the limit. We never have a more complicated asymptotic fragmentation of the whole system. Furthermore, we are able to localize the two limit values: one is strictly smaller than $x_1^* < 1/2$ and the other strictly bigger than $x_2^* > 1/2$, where the points x_i^* depend only on θ and on $(1 - \alpha - \beta)$, which is the 'weight' of the personal inclination component of $P_{n,h}$ in (1). Moreover, the following inequality might provide restrictions on the possible values for N_1:

$$-(1 - \alpha) + (1 - \alpha - \beta) f'(\tilde{z}_{\infty,2}) < \alpha \frac{N_1}{N} < 1 - (1 - \alpha - \beta) f'(\tilde{z}_{\infty,1}).$$

In the following remark we discuss the possible asymptotic synchronization of the system towards the value $1/2$.

Remark 13. (*Possible asymptotic synchronization towards* $1/2$.) In the setting described in Subsection 1.2, the random variable $Z_{n,h}$ represents the present share of agents in market h who have adopted technology 1, and so almost sure asymptotic synchronization towards the value $1/2$ means that in the limit the two technologies in all the markets coexist in the proportions $1:1$. With $f = f_{\text{Tech}}$, the point $1/2\mathbf{1}$ is a synchronization zero point if and only if $\beta = 0$ or $q = 1/2$. Moreover, by Remark 12, if $(1/2)\mathbf{1}$ is a zero point of $\mathbf{F}$, then it is stable (and so a possible limit point for the system) if and only if (U1) is satisfied.

As a consequence of the above results and remarks, arguing as in the proof of Corollary 1, we obtain the following corollary, which deals with the special case $\beta = 0$ or $q = 1/2$.

Corollary 2. (Special case: $\beta = 0$ or $q = 1/2$.) *Take* $f = f_{\text{Tech}}$ *and suppose that one of the conditions* $\beta = 0$ *or* $q = 1/2$ *is satisfied. Then, using the same notation as in Lemma 2 and Theorem 8, only the following cases are possible:*

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(a) *We have* $3\theta - 3/2 \le (1-\alpha)/(1-\alpha-\beta)$. *If this is the case, the system almost surely asymptotically synchronizes and it is predictable, and the unique limit point is* $x^* = 1/2$.

(b) *We have* $(1-\alpha)/(1-\alpha-\beta) < 3\theta - 3/2 \le 1/(1-\alpha-\beta)$. *If this is the case, the system almost surely synchronizes, but there are two possible limit points,* $\mathbf{z}_i^* = z_i^*\mathbf{1}$, $i = 1, 2$, *with* $0 < z_1^* < \widehat{x}_1 < 1/2 < \widehat{x}_2 < z_2^* = 1 - z_1^* < 1$.

(c) *We have* $3\theta - 3/2 > 1/(1-\alpha-\beta)$. *If this the case, the system almost surely converges to a random variable* $\mathbf{Z}_\infty$ *taking values in the set* $\mathcal{SZ}(\mathbf{F})$ *of the stable zero points of* $\mathbf{F}$, *which is contained in* $(0, 1)^N$. *This set always contains two stable synchronization zero points,* $\mathbf{z}_i^* = z_i^*\mathbf{1}$, $i = 1, 2$, *with* $0 < z_1^* < \widehat{x}_1 < 1/2 < \widehat{x}_2 < z_2^* = 1 - z_1^* < 1$, *and any* $\mathbf{z}_\infty \in \mathcal{SZ}(\mathbf{F})$ *which is not a synchronization point has the form* (45), *up to permutations. In particular, when*

$$(1-\alpha-\beta)f'\left(x_1^*\right) \le 1-\alpha,$$

the points of the form (45) *with* $0 < \tilde{z}_{\infty,1} < x_1^* < 1/2$, $1/2 < x_2^* < \tilde{z}_{\infty,2} = 1 - \tilde{z}_{\infty,1} < 1$, $N_1 = N/2$, *and* $(1-\alpha-\beta)f(\tilde{z}_{\infty,1}) - \tilde{z}_{\infty,1} = -(\alpha+\beta)/2$ *are stable non-synchronization zero points of* $\mathbf{F}$.

We conclude this section with two remarks: one regarding the case $N = 1$ and the other the rate of convergence.

Remark 14. (*Case* $N = 1$.) This remark is devoted to the case $N = 1$ and its relationship with the results obtained in [14, 40]. Indeed, we need only consider Theorem 1, Lemma 2, and Remark 12 (with $N = 1$ and $\alpha = \beta = 0$, which corresponds to the case studied in [14, 40]). As a consequence, when $1/2 < \theta \le 5/6$ the system is predictable and the unique limiting configuration is $1/2$. Otherwise it almost surely converges, but it is not predictable, and the two possible limiting configurations z_1^*, z_2^* belong to $(0, 1) \setminus \{1/2\}$ and satisfy $0 < z_1^* < \widehat{x}_1 < 1/2$ and $1/2 < \widehat{x}_2 = 1 - \widehat{x}_1 < z_2^* = 1 - z_1^* < 1$.

Remark 15. (*Rate of convergence.*) When the system is predictable with $\mathbf{z}_\infty = z_\infty\mathbf{1}$ as the unique possible limit value for $\mathbf{Z}_\infty$, applying the same arguments used in Remark 4, we can obtain a central limit theorem where the rate of convergence is driven by $\lambda = (1-\alpha) - (1-\alpha-\beta)f'(z_\infty)$ (see Theorem 13 and Remark 16).

When the system almost surely converges to $\mathbf{Z}_\infty$, but it is not predictable, applying the same arguments used in Remark 11, we get $1/\sqrt{n}$ as the rate of convergence, for any $\mathbf{z}_\infty$ with $P(\mathbf{Z}_n \to \mathbf{z}_\infty) > 0$ and $\lambda(\mathbf{z}_\infty) > 0$ (see Remark 17). Note that when $\mathbf{z}_\infty$ is a possible non-synchronization limit point with $\tilde{z}_{\infty,2} = 1 - \tilde{z}_{\infty,1}$, we have $f'(\tilde{z}_{\infty,1}) = f'(\tilde{z}_{\infty,2})$, and so again $\lambda(\mathbf{z}_\infty) = (1-\alpha) - (1-\alpha-\beta)f'(\tilde{z}_{\infty,1})$.

4. Simulations and figures

In the following section, we consider some graphical illustrations and numerical simulations or samplings of the stochastic dynamical systems. These can easily be coded thanks to the iterative equations defining the dynamical evolution.

We have chosen particular parameter sets for each specific f considered previously. The sets were chosen for their own interest or for their interest in relation to other sets. We used different values for N. We considered either deterministic initial conditions or random ones. When random, we chose independent values, uniformly distributed on $[0, 1]$. Note that when we assume as initial condition $(Z_{0,h}, \ldots, Z_{N,h})$ exchangeable, the variables $\left(Z_{n,1}, \ldots, Z_{n,N}\right)$ are exchangeable for all n, and so the set $\mathcal{Z}$ where $\mathbf{Z}_\infty$ takes values is permutation-invariant,

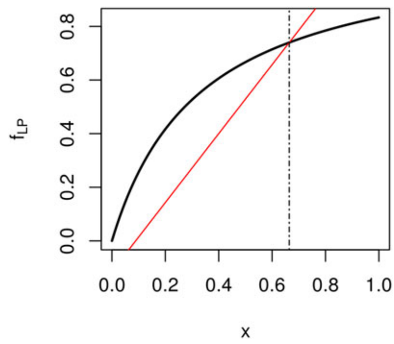

FIGURE 2. Case $f = f_{LP}$. Graph of the function f_{LP} intersecting the straight line $y = ((1-\alpha)x - \beta q)/(1 - \alpha - \beta)$, giving a unique synchronization zero point at ≈ 0.664. Set of parameters: $\theta = 0.9$, $x^* = 1/3$, $\alpha = 0.1$, $\beta = 0.2$, $q = 0.4$.

i.e. if $(z_1, \ldots, z_N)^\top$ belongs to $\mathcal{Z}$, then, for any permutation σ of $\{1, \ldots, N\}$, the vector $(z_{\sigma(1)}, \ldots, z_{\sigma(N)})^\top$ also belongs to $\mathcal{Z}$.

Since f is nonlinear, unlike in the case of the models where f is linear, combinatorics can create multiple limit points. For $f = f_{LP}$, only synchronization limit points are possible. For $f = f_{LogP}$ or $f = f_{\mathrm{Tech}}$, depending on the choice of parameters, many limit points are possible, and these can be of synchronization type (on the diagonal) or of non-synchronization type (off the diagonal). When f is linear, synchronization has been proved to hold as soon as there is interaction ($\alpha > 0$). Here, by contrast, specific conditions between the parameters must be fulfilled to guarantee synchronization almost surely. Finally, as observed in the following samplings, in some region of parameters, limit points may be difficult to observe computationally because of slow dynamical evolution.

4.1. Case $f = f_{LP}$

For the set of parameters $\theta = 0.9$, $x^* = 1/3$, $\alpha = 0.1$, $\beta = 0.2$, $q = 0.4$, Figure 2 shows the graph of the function f_{LP} intersecting the straight line defined through (15). There is a unique synchronization zero point at ≈ 0.664.

For the same set of parameters, Figure 3 shows some numerical simulation samples. Figure 3(a) presents the trajectories of the component values of one sample of the whole system, when $N = 30$. The associated empirical means are represented in Figure 3(b). In both cases, almost sure convergence towards the unique (stable) synchronization point is observed, in coherence with the previously stated theoretical result. Figure 3(c) illustrates with a histogram the values observed for a large time ($T = 5000$). Notice that these values are component values of 100 independent samples of the system. Figure 3(d) is a representation when $N = 2$ of the tangent/gradient field of $-V$. Furthermore, values of $-V$ are represented through colors. Blue is used for low values and shows a unique minimum of $-V$. Red is used for high values.

4.2. Case $f = f_{\mathrm{LogP}}$

In the following figures, certain specific sets of parameters were chosen.

In Figure 4, parameters are chosen such that there is a unique stable synchronization zero point. Figure 4(a) illustrates, for one sample, the almost sure convergence towards this value for $(Z_n(k))_n$, and Figure 4(b) shows the trajectories of the associated empirical means $(I_n(k))_n$. Figure 4(c) shows a histogram of the component values $Z_n(k)$ for n large and $k \in \{1, \ldots, N\}$.

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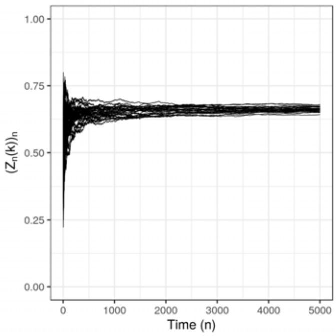
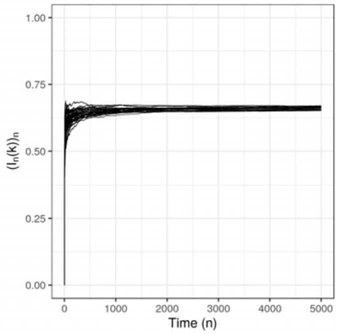
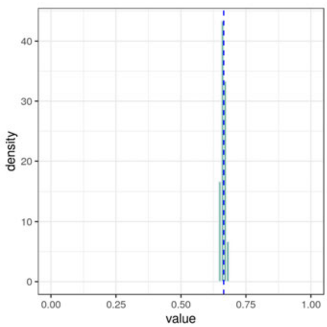


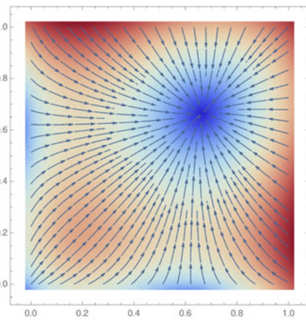
(a) One sample of the trajectories $(Z_n(k))_n$ $(1 \leq k \leq N)$ of a system with $N = 30$. Starting condition is $1/2$ for all components.



(b) One sample of the trajectories of the associated empirical means $(I_n(k))_n$ $(1 \leq k \leq N)$.



(c) Histogram of system's $(N = 30)$ components values from 100 (independent) samples, at time 5000. Starting conditions are always $1/2$.



(d) Representation of the field $F = -\nabla V$ when $N = 2$ computed using the software Mathematica 11.3.

FIGURE 3. Case $f = f_{LP}$. Set of parameters is $\theta = 0.9$, $x^* = 1/3$, $\alpha = 0.1$, $\beta = 0.2$, $q = 0.4$. There is a unique (stable) synchronization point at ≈ 0.664. The system is predictable.

There were 100 independent samples of the whole system. Note that the $(N = 15) * 100$ values used for the histogram are not independent. As previously, Figure 4(d) represents, when $N = 2$, the 'landscape' of $F = -\nabla V$.

In Figure 5, two parameters sets are considered: the one from Figure 9 and the one from Figure 6, Figure 7, and Figure 8. For illustration, stable synchronization zero points are found at the intersection of the curve associated to f and the straight line given by Equation (15). In both cases, there are two stable synchronization zero points and some non-synchronization stable zero points. In Figure 9 the component values are different, unlike the case for the set of parameters from Figure 6, Figure 7, and Figure 8, where the component values for synchronization points and non-synchronization points are close.

The set of parameters in Figure 9 is related to Corollary 1.

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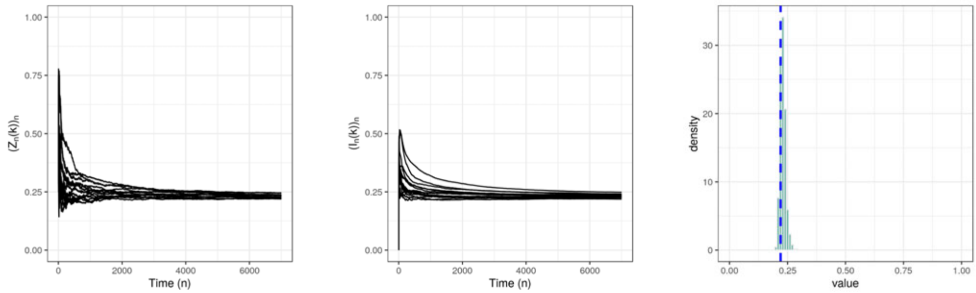
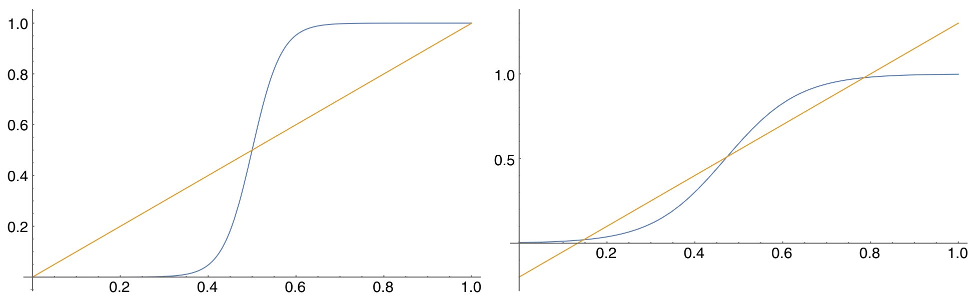


(a) One sample of the trajectories $(Z_n(k))_n$ $(1 \leq k \leq N)$.

(b) One sample of the trajectories of the associated empirical means $(I_n(k))_n$ $(1 \leq k \leq N)$.

(c) Histogram of system's components values, for 100 independent samples, at time 7000.

FIGURE 4. Case $f = f_{LogP}$. Parameters are $x^\star = 0.6$, $\theta = 5$, $\alpha = 0.1$, $\beta = 0.3$, $q = 0.4$. There is a unique zero of F and synchronization point at ≈ 0.22 (vertical dashed line in Panel (c)). System size is $N = 15$. Starting condition is 1/2 for all components.



(a) Set of parameters $\theta = 30$, $x^* = 1/2$, $\alpha = 0.4$, $\beta = 0$, $q = 0$ as in Fig. 9. It gives two stable zero synchronisation points close to 0 and close to 1 at the intersection of both curves

(b) Set of parameters $\theta = 12$, $x^* = 0.47$, $\alpha = 0.1$, $\beta = 0.3$, $q = 0.4$ as in Fig. 6. It gives two stable zero synchronisation points at the intersection of both curves.

FIGURE 5. Case $f = f_{LogP}$. Graph of the function f_{LogP} in blue intersecting the straight line (orange) defined through $y = ((1-\alpha)x - \beta q)/(1-\alpha-\beta)$ from Equation (15).

In Figure 6, the initial condition is always 1/2 and $N = 30$ was chosen. In Figure 7 the initial conditions are independent and uniformly distributed. The case $N = 5$ is considered differently from Figure 8, where $N = 30$ is chosen.

As can be deduced from Figure 7(b) and Figure 8(b), the convergence is not always towards the same zero point. Non-synchronization zero points are observed as limits. It is possible that synchronization zero points are rarely observed, since Figure 8(b) and Figure 8(c) show no observation going to the synchronization values. For large values of N it seems that

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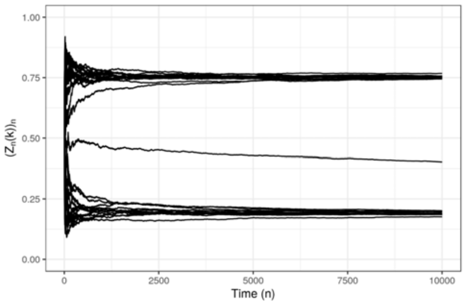
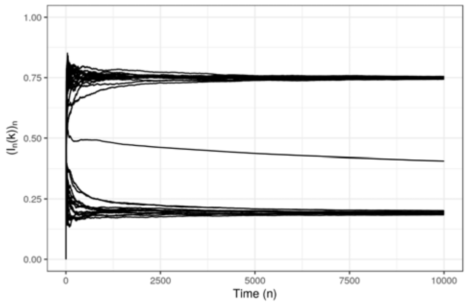
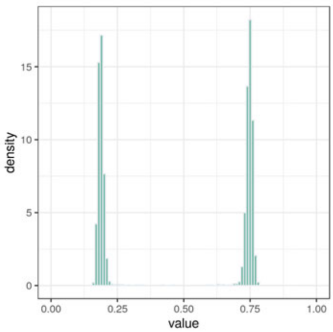


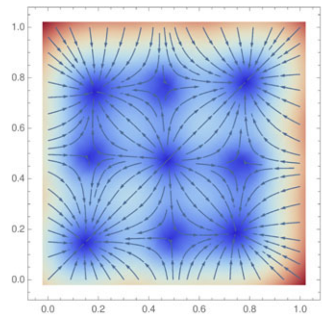
(a) One sample of the trajectories $(Z_n(k))_n$ $(1 \leq k \leq N)$. $N = 30$.



(b) One sample of the trajectories of the associated empirical means $(I_n(k))_n$ $(1 \leq k \leq N)$.



(c) $N = 30$. Histogram of components' values at $T = 10000$ iterations. Independent system's sample of size 100.



(d) Field $F = -\nabla V$ associated when $N = 2$.

FIGURE 6. Case $f = f_{LogP}$. Parameters are $x^\star = 0.47$, $\theta = 12$, $\alpha = 0.1$, $\beta = 0.3$, $q = 0.4$. There are two stable synchronization zeros $\approx \{0.14,\ 0.78\}$. Component values of (stable) non-synchronization points are close to 0.2 and 0.8. Starting conditions are 1/2 for every component and sample.

synchronization is never observed (except with very specific starting conditions close to the synchronization points).

In Figure 9, the 'landscape' associated to this parameter set when $N = 2$ is shown in Panel (d). Convergence towards non-synchonization points is observed in particular in the sample in Figure 9(a) with $N = 100$. Figure 9(c) depicts trajectories represented in the potential landscape when $N = 2$.

4.3. Case $f = f_{\mathrm{Tech}}$

The parameters for Figure 10 are such that there are two stable synchronization zero points and there exist stable non-synchronization zero points. As can be observed from simulations in Figure 10(c) and in the landscape in Figure 10(d), the dynamics can be very slow close to these points. Samples from Figure 10(a) support this observation.

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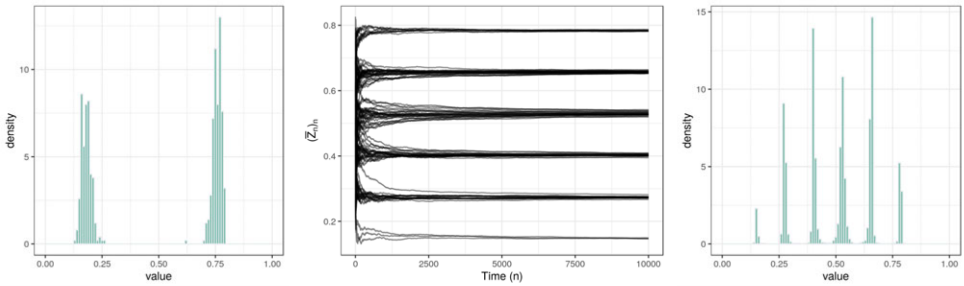
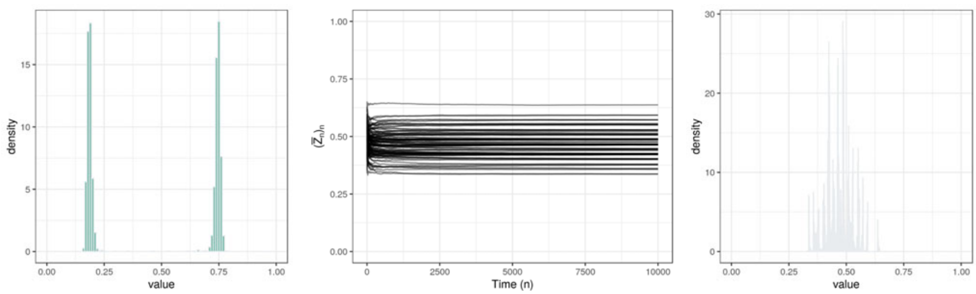


(a) Histogram of component's values at time 10000.

(b) Trajectories of mean field for each system sampled. Thus these are 100 independent curves.

(c) Histogram of 100 independent mean field's values at time 10000

FIGURE 7. Case $f = f_{LogP}$. Same set of parameters as Figure 6: $x^{\star} = 0.47$, $\theta = 12$, $\alpha = 0.1$, $\beta = 0.3$, $q = 0.4$. Here $N = 5$, and starting conditions are chosen independently and uniformly distributed on [0, 1]. Panels (a), (b), and (c) are related to the same 100 independent system samples.



(a) Histogram of components' values at time 10000

(b) Trajectories of mean field from 100 independent samples.

(c) Histogram of the mean field's values at time 10000, from a set of 100 independent system's samples.

FIGURE 8. Case $f = f_{LogP}$. Parameters are as in Figure 6 and Figure 7 but with uniformly distributed starting conditions. Here, $N = 30$.

The parameters for Figure 11 are such that there are two stable synchronization zero points and there are no stable non-synchronization zero points. In Figure 11(c) it can be seen for these samples that the dynamical behavior is slow in the neighborhood of the unstable non-synchronization points. Contrary to what is observed (because the number of iterations is finite), we know from the previously mentioned theoretical results that convergence towards stable synchronization points will eventually happen.

In Figure 12, the parameters are set up so that there are only two stable synchronization points. From $N = 2$ cases, we can guess that there are unstable non-synchronization points in regions where the dynamics are slow. For instance, in Figure 12(b), the sample 6 starts at

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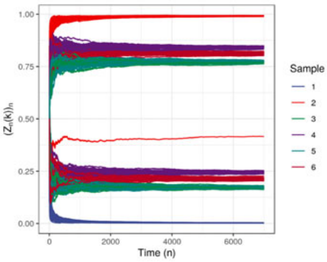
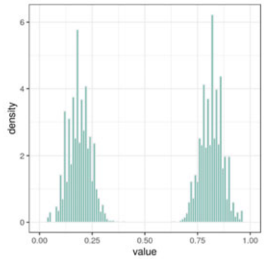
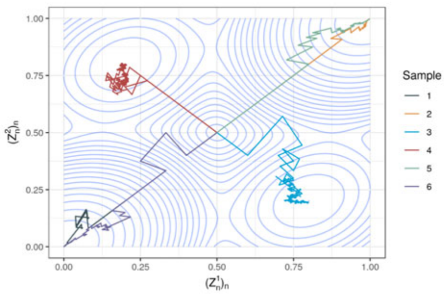


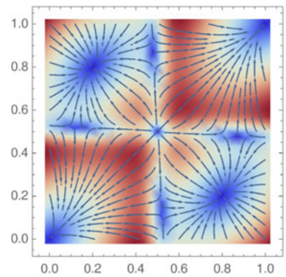
(a) Six samples of the system $(Z_n(k))_n$ $(1 \le k \le N)$ with $N = 100$. Sample 1 starts with 0.2 on the diagonal. Sample 2 starts with 0.7 on the diagonal. Samples 3 to 6 starts with 0.5 on the diagonal. All trajectories of one system's sample share the same color.



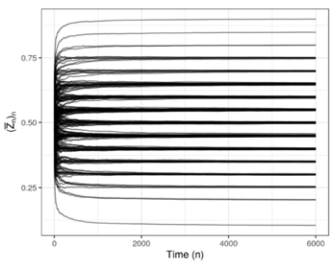
(b) Histogram of components' values at $T = 6000$ for 200 samples of the systems with $N = 20$. For each component of each sample, the starting condition is chosen, independetly, uniformly distributed on $[0, 1]$.



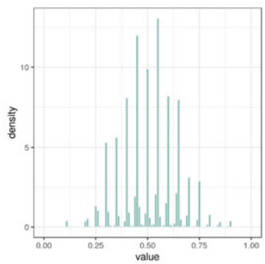
(c) Six samples of the system when $N = 2$. Representation of trajectories $(Z_n(1), Z_n(2))_n$ up to 30000 iterations. Each color means a different sample. Sample 1 starts with $(0.2, 0.2)$. Sample 2 starts with $(0.7, 0.7)$. Samples 3 to 6 start with 0.5 on the diagonal. In background some level sets of $-V$.



(d) Field $F = -\nabla V$ when $N = 2$.



(e) Mean fields trajectories associated to (B).



(f) Histogram of mean field values at $T = 6000$ for 200 independent samples of the system with $N = 20$. Same sample as in (B) and (E)

FIGURE 9. Case $f = f_{LogP}$. Parameters are $x^\star = 0.5$, $\theta = 30$, $\alpha = 0.4$, $\beta = 0$. This is related to Corollary 1. Synchronization points are close to 0 and 1 (stable) and $x^* = 0.5$ (unstable). Component values of non-synchronization (stable) points are close to 0.2 and 0.8.

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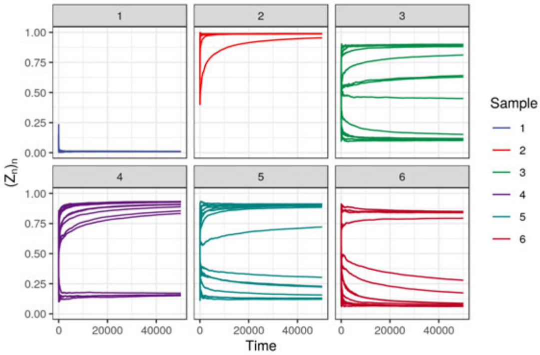
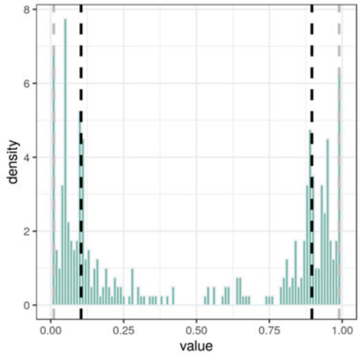
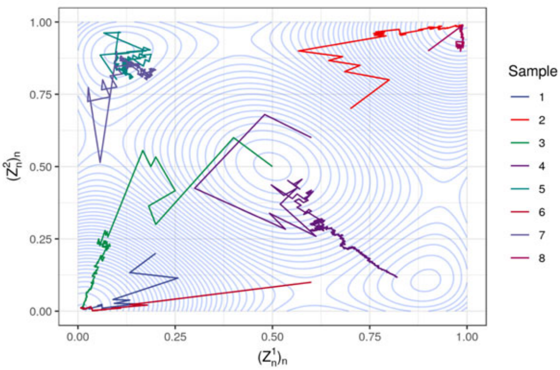


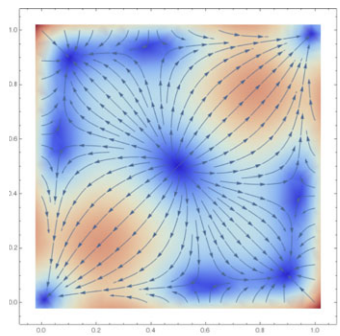
(a) Six samples of the system's trajectories when $N = 15$. Each component starts at 0.5 on the diagonal.



(b) Histogram of components' values at time $T = 50000$ when $N = 4$. All trajectories start in $1/2$. Vertical grey dashed lines are at stable synchronization points values 0.010 and 0.989. Vertical black dashed lines are at non-synchronization values 0.104, and 0.896.

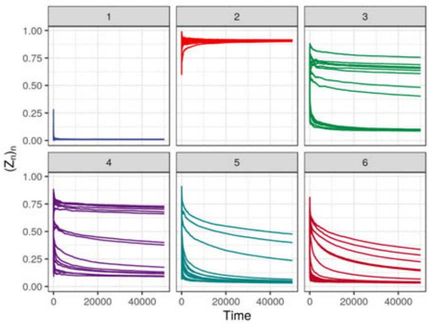


(c) Eight samples of the systems when $N = 2$. Representation of trajectories $(Z_n(1), Z_n(2))_n$. Each color means a different sample. Sample 1 starts with $(0.2, 0.2)$. Sample 2 starts with $(0.7, 0.7)$. Samples 3 and 4 start with $(0.5, 0.5)$. Sample 5 starts with $(0.1, 0.8)$. Sample 6 starts with $(0.6, 0.1)$. Sample 7 starts with $(0.1, 0.9)$. Sample 8 starts with $(0.4, 0.95)$. In background some level sets of the associated $-V$. Time up to 20000 iterations.

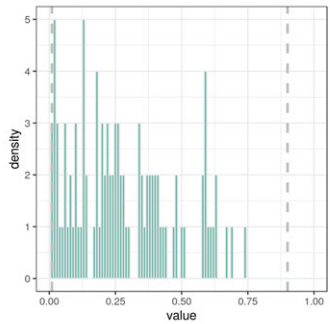


(d) Representation of the field $F = -\nabla V$ when $N = 2$.

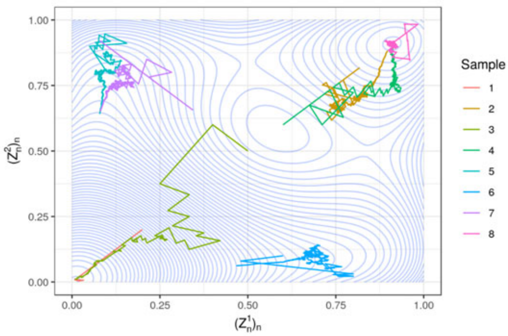
FIGURE 10. Case $f = f_{\text{Tech}}$. Parameters are $\theta = 0.99$, $\alpha = 0.14$, $\beta = 0$. There are two stable synchronization points $\{\approx 0.0103, \ \approx 0.989\}$, and there are stable non-synchronization points. Component values of non-synchronization zero points belong to $\{\approx 0.104, \ \approx 0.896\}$.



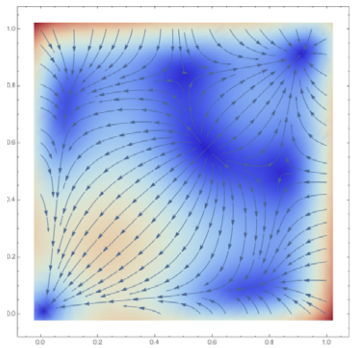
(a) Six samples of the system's trajectories when $N = 15$. Each system starts at $1/2$ on the diagonal.



(b) Histogram of mean field's final values, at $T = 50000$, when $N = 10$, sample of size 100, uniformly distributed initial condition.



(c) Eight samples of the systems when $N = 2$. Representation of trajectories $(Z_n(1), Z_n(2))_n$. Each color means a different sample. Sample 1 starts with $(0.2, 0.2)$. Sample 2 starts with $(0.7, 0.7)$. Samples 3 starts with $(0.5, 0.5)$. Samples 4 starts with $(0.6, 0.6)$. Sample 5 starts with $(0.1, 0.8)$. Sample 6 starts with $(0.6, 0.1)$. Sample 7 starts with $(0.1, 0.9)$. Sample 8 starts with $(0.9, 0.9)$. In background some level sets of the associated $-V$. Time up to 200000 iterations.



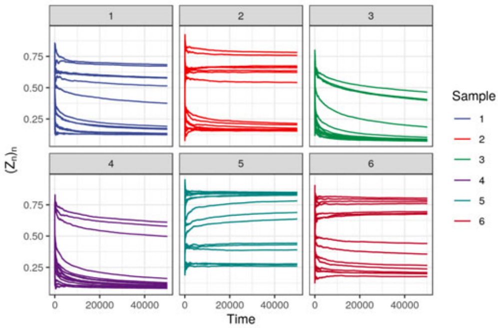
(d) Representation of the field when $N = 2$.

FIGURE 11. Case $f = f_{\text{Tech}}$. Parameters are $\theta = 0.99$, $\alpha = 0.15$, $\beta = 0.05$, $q = 0.005$. There are only two stable synchronization zero points. Non-synchronization points exist but are unstable.

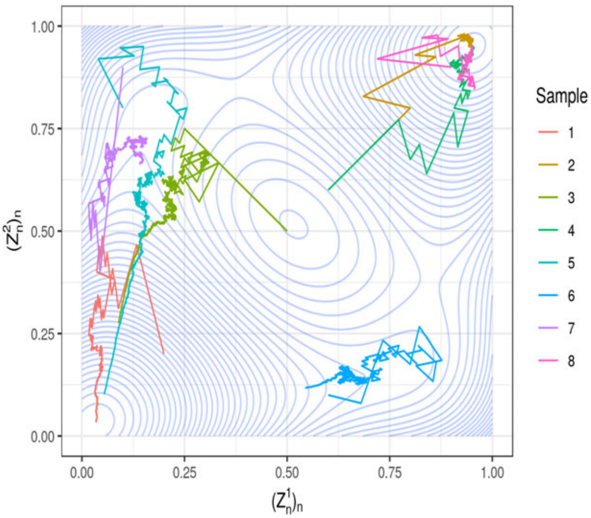
(0.6, 0.1) and does not succeed in reaching the neighborhood of a synchronization point before 150,000 iterations.

Appendix A. Stochastic approximation

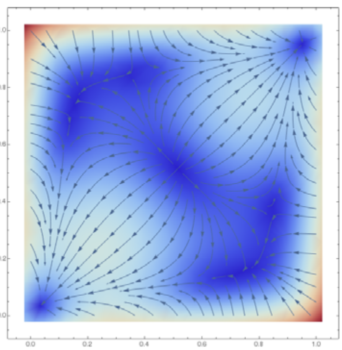
Here, we briefly recall the results from stochastic approximation theory used in the present work. We refer the interested reader to more complete monographs (e.g. [17, 25, 38, 43, 44, 61, 62, 75]).



(a) Six samples of the system's trajectories when $N = 10$. Each component start at $1/2$.



(b) Six samples of the systems when $N = 2$. Representation of trajectories $(Z_n(1), Z_n(2))_n$. Time up to 150000 iterations. Each color means a different sample. Sample 1 starts with 0.2 on the diagonal. Sample 2 starts with 0.7 on the diagonal. Samples 3 to 8 start diversely, for instance sample 6 starts at $(0.6, 0.1)$. In background some level sets of the associated $-V$.



(c) Representation of the field $F = -\nabla V$ when $N = 2$.

FIGURE 12. Case $f = f_{\text{Tech}}$. Parameters are $\theta = 0.97$, $\alpha = 0.18$, $\beta = 0.001$, $q = 0.01$. There are only two stable synchronization zero points.

Let $\mathbf{Z} = (\mathbf{Z}_n)_{n\geq 0}$ be an N-dimensional stochastic process with values in $[0, 1]^N$, adapted to a filtration $\mathcal{F} = (\mathcal{F}_n)_{n\geq 0}$. Suppose that $\mathbf{Z}$ satisfies

$$\mathbf{Z}_{n+1} = \mathbf{Z}_n + r_n\mathbf{F}(\mathbf{Z}_n) + r_n\Delta\mathbf{M}_{n+1}\ , \tag{46}$$

where $r_n \sim 1/n$, $\mathbf{F}$ is a bounded $\mathcal{C}^1$ vector-valued function on an open subset $\mathcal{O}$ of $\mathbb{R}^N$, with $[0, 1]^N \subset \mathcal{O}$, and $(\Delta\mathbf{M}_n)_n$ is a bounded martingale difference with respect to $\mathcal{F}$.

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We have the following results.

Theorem 9. (E.g. [62, 63].) *The limit set of* $\mathbf{Z}$*, i.e. the set defined as* $\mathcal{L}(\mathbf{Z}) = \bigcap_{n\geq 0} \overline{\bigcup_{m\geq n} \mathbf{Z}_m}$*, is almost surely a compact connected set that is stable under the flow of the differential equation* $\dot{\mathbf{z}} = \mathbf{F}(\mathbf{z})$*.*

Therefore, the asymptotic behavior of the stochastic process $\mathbf{Z}$ is related to the properties of the zero points of the vector field $\mathbf{F}$. In the next definition, we give a classification of these points.

Definition 1. A zero point of $\mathbf{F}$ is a point $\mathbf{z}$ such that $\mathbf{F}(\mathbf{z}) = \mathbf{0}$. We denote by $\mathcal{Z}(\mathbf{F})$ the set of all zero points of $\mathbf{F}$. Moreover, denoting by $J(\mathbf{F})(\mathbf{z})$ the Jacobian matrix of $\mathbf{F}$ computed at the point $\mathbf{z}$, we classify the zero points of $\mathbf{F}$ according to the sign of the real part of the eigenvalues of $J(\mathbf{F})(\mathbf{z})$ as follows:

- $\mathbf{x}$ is called a *stable* zero point if all the eigenvalues of $J(\mathbf{F})(\mathbf{z})$ have negative (meaning 'not strictly positive', that is, ≤ 0) real part;
- $\mathbf{x}$ is called a *strictly stable* zero point if all the eigenvalues of $J(\mathbf{F})(\mathbf{z})$ have strictly negative real part;
- $\mathbf{x}$ is called a *linearly unstable* (or unstable) zero point if $J(\mathbf{F})(\mathbf{z})$ has at least one eigenvalue with strictly positive real part;
- $\mathbf{x}$ is called a *repulsive* zero point if all the eigenvalues of $J(\mathbf{F})(\mathbf{z})$ have strictly positive real part.

Suppose that, for each $\mathbf{z}$, the Jacobian matrix $J(\mathbf{F})(\mathbf{z})$ is symmetric. Then all its eigenvalues are real, and since the sign of the scalar product $\langle \mathbf{F}(\mathbf{z}') - \mathbf{F}(\mathbf{z}), \mathbf{z}' - \mathbf{z}\rangle$ for $\mathbf{z}'$ in a neighborhood of $\mathbf{z}$ is related to the property of $J(\mathbf{F})(\mathbf{z})$ being positive/negative (semi)definite, and this last property is related to the sign of the eigenvalues of $J(\mathbf{F})(\mathbf{z})$, we can state the following:

- $\mathbf{z}$ is a stable zero point if and only if $\langle \mathbf{F}(\mathbf{z}'), \mathbf{z}' - \mathbf{z}\rangle \leq 0$ for all $\mathbf{z}'$ in a neighborhood of $\mathbf{z}$;
- $\mathbf{z}$ is a linearly unstable zero point if and only if, for any neighborhood $\mathcal{B}_{\mathbf{z}}$ of $\mathbf{z}$, there exists $\mathbf{z}' \in \mathcal{B}_{\mathbf{z}}$ such that $\langle \mathbf{F}(\mathbf{z}'), \mathbf{z}' - \mathbf{z}\rangle > 0$.

Theorem 10. (E.g. [63].) *If there exists a stable zero point* $\mathbf{z}$ *of* $\mathbf{F}$ *such that*

$$\langle \mathbf{F}(\mathbf{Z}_n), \mathbf{Z}_n - \mathbf{z}\rangle < 0 \qquad \forall n \text{ with } \mathbf{Z}_n \neq \mathbf{z},$$

then $\mathbf{Z}_n \xrightarrow{a.s.} \mathbf{z}$*.*

Theorem 11. ([38, Chapter 2, Theorem 2], [61, Chapter 5, Theorem 2.1], or [75, Theorem 2.18].) *If* $\mathbf{F} = -\nabla \mathbf{V}$ *and* $\mathcal{Z}(\mathbf{F})$ *is non-empty and finite, then there exists a random variable* $\mathbf{Z}_\infty$ *which takes values in* $\mathcal{Z}(\mathbf{F})$ *and such that*

$$\mathbf{Z}_n \xrightarrow{a.s.} \mathbf{Z}_\infty.$$

Theorem 12. ([17, Theorem 9.1] or [74, Theorem 1].) *If* $\mathbf{F} \in \mathcal{C}^2$ *and there exists a constant* $C > 0$ *such that we have*

$$E\big[\langle \Delta \mathbf{M}_{n+1}, \mathbf{v}\rangle)^+ | \mathcal{F}_n\big] \geq C \qquad \forall \mathbf{v} \in \mathbb{R}^N \text{ with } |v| = \sum_{h=1}^{N} |v_h| = 1, \tag{47}$$

then, for each linearly unstable zero point $\mathbf{z}$ *of* $\mathbf{F}$*, we have* $P(\mathbf{Z}_n \to \mathbf{z}) = 0$*.*

If $\mathbf{F}$ belongs to $\mathcal{C}^2$ and, for each $\mathbf{z}$, the Jacobian matrix $J(\mathbf{F})(\mathbf{z})$ is symmetric (and so all its eigenvalues are real and it is diagonalizable), then from [80] we get the following central limit theorem.

Theorem 13. *Suppose* $\mathbf{F} \in \mathcal{C}^2$, *with* $J(\mathbf{F})(\mathbf{z})$ *symmetric for each* $\mathbf{z}$. *Let* $\mathbf{z}_\infty \in (0, 1)^N$ *be a strictly stable zero point of* $\mathbf{F}$ *such that* $\mathbf{Z}_n \xrightarrow{a.s.} \mathbf{z}_\infty$. *Suppose that*

$$E\left[\Delta\mathbf{M}_{n+1}(\Delta\mathbf{M}_{n+1})^\top | \mathcal{F}_n\right] \xrightarrow{a.s.} \Gamma, \tag{48}$$

where $\Gamma = \Gamma(\mathbf{z}_\infty)$ *is a deterministic symmetric positive definite matrix. Denote by* $\lambda = \lambda(\mathbf{z}_\infty)$ *the smallest eigenvalue of* $-J(\mathbf{F})(\mathbf{z}_\infty)$. *Then we have the following:*

- *If* $\lambda > 1/2$, *then*
$$\sqrt{n}(\mathbf{Z}_n - \mathbf{z}_\infty) \xrightarrow{d} \mathcal{N}(\mathbf{0}, \Sigma),$$
where
$$\Sigma = \Sigma(\mathbf{z}_\infty) = \int_0^{+\infty} e^{\left(J(\mathbf{F})(\mathbf{z}_\infty)+\frac{Id}{2}\right)u}\,\Gamma\, e^{\left(J(\mathbf{F})(\mathbf{z}_\infty)+\frac{Id}{2}\right)u}\, du.$$
- *If* $\lambda = 1/2$, *then*
$$\sqrt{\frac{n}{\ln(n)}}(\mathbf{Z}_n - \mathbf{z}_\infty) \xrightarrow{d} \mathcal{N}(\mathbf{0}, \Sigma),$$
where
$$\Sigma = \Sigma(\mathbf{z}_\infty) = \lim_{n\to+\infty} \frac{1}{\ln(n)} \int_0^{\ln(n)} e^{\left(J(\mathbf{F})(\mathbf{z}_\infty)+\frac{Id}{2}\right)u}\,\Gamma\, e^{\left(J(\mathbf{F})(\mathbf{z}_\infty)+\frac{Id}{2}\right)u}\, du.$$
- *If* $0 < \lambda < 1/2$, *then*
$$n^\lambda(\mathbf{Z}_n - \mathbf{z}_\infty) \xrightarrow{a.s.} V,$$
where V *is a suitable finite random variable.*

Note that Assumption 2.2 in [80] is satisfied since we take $\mathbf{F} \in \mathcal{C}^2$. Equation (2.3) of Assumption 2.3 in [80] holds because we assume $(\Delta\mathbf{M}_n)_n$ bounded. Equation (2.4) of the same assumption is implied by the above condition (48). All the assumptions on the remainder term $\mathbf{r}_n$ in [80] are satisfied because we have $\mathbf{r}_n = \mathbf{0}$. Finally, since $J(\mathbf{F})(\mathbf{z}_\infty)$ is symmetric, the matrix $e^{\left(J(\mathbf{F})(\mathbf{z}_\infty)+\frac{Id}{2}\right)u}$ is also symmetric.

Remark 16. With the same assumptions and notation as in Theorem 13, the limit covariance matrix Σ in the case $\lambda > 1/2$ can be rewritten using the Lyapunov equation (e.g. [53, Theorem 2.1]):

$$\left(J(\mathbf{F})(\mathbf{z}_\infty) + \frac{1}{2}Id\right)\Sigma + \Sigma\left(J(\mathbf{F})(\mathbf{z}_\infty)^\top + \frac{1}{2}Id\right) = -\Gamma.$$

Since $J(\mathbf{F})(\mathbf{z}_\infty)$ is symmetric by assumption and Σ is symmetric by definition, we have

$$\Sigma = (-2J(\mathbf{F})(\mathbf{z}_\infty) - Id)^{-1}\,\Gamma.$$

Remark 17. In [53] we also have a central limit theorem when there exist more limit points for $(\mathbf{Z}_n)_n$. Indeed, under the same assumptions on $\mathbf{F}$ as in Theorem 13, when the condition (48) is satisfied and $\mathbf{z}_\infty \in (0, 1)^N$ is a strictly stable zero point of $\mathbf{F}$ such that $P(\mathbf{Z}_n \to \mathbf{z}_\infty) > 0$ and $\lambda(\mathbf{z}_\infty) > 1/2$, we have to consider the convergence in distribution under the probability

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measure $P(\cdot|\mathbf{Z}_n \to \mathbf{z}_\infty)$, and the corresponding limit distribution is the one with characteristic function

$$\mathbf{u} \mapsto E\left[\exp\Big(-\frac{1}{2}\mathbf{u}^\top \Sigma(\mathbf{z}_\infty)\mathbf{u}\Big) \mid \mathbf{Z}_n \to \mathbf{z}_\infty\right],$$

where $\Sigma(\mathbf{z}_\infty)$ is defined as in Theorem 13 or, equivalently, as in Remark 16.

Appendix B. Eigenvalues of the Jacobian matrix

We observe that, letting $d_i := (1-\alpha-\beta)f'(z_i) - 1$ for $i = 1, \ldots, N$, the Jacobian matrix of $\mathbf{F}$ is given by

$$J\mathbf{F}(\mathbf{z}) = \frac{\alpha}{N}\begin{pmatrix} 1 & \ldots & 1 \\ \vdots & \ldots & \vdots \\ 1 & \ldots & 1 \end{pmatrix} + diag(d_1, \ldots, d_N)\ , \tag{49}$$

where $diag(d_1, \ldots, d_N)$ denotes the diagonal matrix with diagonal elements $d_1, \ldots, d_N$.

In order to compute its eigenvalues, we use the following results.

Lemma 3. *Assume that the matrix A has the form*

$$A = c^2\begin{pmatrix} 1 & \ldots & 1 \\ \vdots & \ldots & \vdots \\ 1 & \ldots & 1 \end{pmatrix} + diag(d_1, \ldots, d_N),$$

where $c > 0$ and $diag(d_1, \ldots, d_N)$ denotes the diagonal matrix with diagonal elements equal to d_i for $i = 1, \ldots, N$. Then the characteristic polynomial of A can be written as

$$p(\lambda) = \prod_{i=1}^N (d_i - \lambda) + c^2 \sum_{i=1}^N \prod_{j\neq i} (d_j - \lambda). \tag{50}$$

Proof. Set $\mathbf{v} = (c, \ldots, c)^\top$. By the matrix determinant lemma, we get for all $\lambda \notin \{d_1, \ldots, d_N\}$

$$p(\lambda) = det(A - \lambda I) = det(c^2\mathbf{v}\mathbf{v}^\top + diag(d_1 - \lambda, \ldots, d_N - \lambda))$$
$$= \left(1 + c^2\sum_{i=1}^N \frac{1}{d_i - \lambda}\right)\prod_{i=1}^N (d_i - \lambda) = \prod_{i=1}^N (d_i - \lambda) + c^2\sum_{i=1}^N \prod_{j\neq i}(d_j - \lambda).$$

By continuity, we can conclude that the above expression for $p(\lambda)$ holds for all λ. □

Corollary 3. *With the same assumptions and notation as in the above lemma, the number d_k is an eigenvalue of $J\mathbf{F}(\mathbf{z})$ if and only if there exists at least one $j \neq k$ such that $d_j = d_k$.*

Proof. Clearly, if $d_k = d_j$ for at least one pair $k \neq j$ we have $p(d_k) = 0$. On the other hand, if $p(d_k) = 0$ we necessarily have $\prod_{j\neq k}(d_j - d_k) = 0$, which implies $d_j = d_k$ for at least one $j \neq k$. □

Corollary 4. *With the same assumption and notation as in the above lemma, if $d_i = d$ for $i = 1, \ldots, N$, then the eigenvalues of A are*

$$\lambda_1 = d \qquad \textit{and} \qquad \lambda_2 = d + c^2 N.$$

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Proof. In this case, we have

$$p(\lambda) = (d - \lambda)^{N-1}\big(d - \lambda + c^2 N\big),$$

and so d is an eigenvalue with multiplicity $N-1$ and $d + c^2N$ is an eigenvalue with multiplicity 1. □

Corollary 5. *With the same assumptions and notation as in the above lemma, suppose that $d_i \in \{D_1, D_2\}$, with $D_1 \neq D_2$, for all $i = 1, \ldots, N$, and assume that $d_i \neq d_j$ for at least one pair of different indices. Moreover, denote by $N_1 \in \{1, \ldots, N-1\}$ the number of indices such that $d_i = D_1$ and $N_2 = N - N_1$. The eigenvalues of $\mathbf{JF}(\mathbf{z})$ are as follows:*

- $\lambda = D_1$ *with multiplicity* $N_1 - 1$*;*
- $\lambda = D_2$ *with multiplicity* $N_2 - 1$*;*
- $\lambda = \lambda_i$ *with* $i = 1, 2$*, where the* λ_i *are the solutions of the equation*

$$\lambda^2 - \big(D_1 + D_2 + c^2N\big)\lambda + D_1D_2 + c^2N_1D_2 + c^2N_2D_1 = 0. \tag{51}$$

Then, in particular, we have the following:

(a) If $D_1 \geq 0$ and $D_2 \geq 0$, then all the eigenvalues are positive.

(b) If $D_1 = 0$ and $D_2 < 0$ (or vice versa), then there exists a strictly positive eigenvalue.

(c) When $D_1 < 0$, $D_2 < 0$, all the eigenvalues are negative if and only if we have

$$D_1 + D_2 + c^2N < 0 \qquad \textit{and} \qquad D_1D_2 + c^2N_1D_2 + c^2N_2D_1 \geq 0. \tag{52}$$

Proof. We first observe that the matrix A is symmetric, and so all its eigenvalues are real numbers; this, together with the formula (50), proves the first assertion. Let us write the polynomial in (51) as $r(\lambda) = \lambda^2 - B\lambda + C$, where $B = D_1 + D_2 + c^2N$ and $C = D_1D_2 + c^2N_1D_2 + c^2N_2D_1$. For the statement (a), observe that if $D_1, D_2 \geq 0$, then $B > 0$ and $C \geq 0$, and this implies that the zeros of $r(\lambda)$ are both positive. Similarly, in the case (b), if $D_1 = 0$, $D_2 < 0$ (or vice versa), we have $C < 0$ and so one of the zeros of $r(\lambda)$ must be strictly positive. Finally, in the case (c), it is enough to observe that the zeros of $r(\lambda)$ are both negative if and only if $B < 0$ and $C \geq 0$. □

Remark 18. A necessary condition for (52) is

$$D_i < -c^2N_i \quad \text{for } i = 1, 2, \tag{53}$$

while a sufficient condition is

$$D_i \leq -c^2N_i(1 + \delta_i) \quad \text{for } i = 1, 2, \tag{54}$$

with $\delta_1,\ \delta_2 > 0$ and $\delta_1\delta_2 \geq 1$.

Indeed, observe that the second condition in (52) can be written as $\big(D_1 + c^2N_1\big)\big(D_2 + c^2N_2\big) - c^4N_1N_2 \geq 0$, which implies $\big(D_1 + c^2N_1\big)\big(D_2 + c^2N_2\big) > 0$ (because $c,\ N_1,\ N_2 > 0$). Thus we have either $D_i < -c^2N_i$ for both $i = 1, 2$ or $D_i > -c^2N_i$ for both $i = 1, 2$, and the first equation in (52) excludes the second case. Hence we necessarily have $D_i < -c^2N_i$ for $i = 1, 2$. On the other hand, a simple computation shows that the condition (54) implies (52): we have

$$D_1 + D_2 + c^2N \leq -c^2\big(N_1\delta_1 + N_2\delta_2\big) < 0$$

and

$$\big(D_1 + c^2 N_1\big)\big(D_2 + c^2 N_2\big) - c^4 N_1 N_2 = \big(-D_1 - c^2 N_1\big)\big(-D_2 - c^2 N_2\big) - c^4 N_1 N_2$$
$$\geq c^4 N_1 N_2 (\delta_1 \delta_2 - 1) \geq 0.$$

Remark 19. If in (54) we take $N_1(1+\delta_1) = N_2(1+\delta_2) = N$ (that is $\delta_i = NN_i - 1 \geq 1$), we obtain

$$D_i \leq -c^2 N \quad \forall i = 1, 2.$$

Appendix C. Zero points of polynomial systems

Lemma 4. *Assume f is a real polynomial of degree $d \geq 2$ such that $F = (F_1, \ldots, F_N) : [0, 1]^N \to [0, 1]^N$ ($N \geq 1$) with*

$$F_i(z) = \alpha \bar{z} + \beta q + (1 - \alpha - \beta) f(z_i) - z_i,$$

where $(\alpha, \beta) \in [0, 1]^2$ and $1 - \alpha - \beta \neq 0$. Then the set $\mathcal{Z}(\mathbf{F})$ of zero points of the system $(F_1, \ldots, F_N)$ is finite.

Proof. Let $\mathcal{Z}_{\mathbb{C}}(\mathbf{F})$ be the algebraic set of the solutions $z \in \mathbb{C}$ such that for all i with $1 \leq i \leq N$, $F_i(z) = 0$. Let I be the ideal generated by the polynomials $(F_1, \ldots, F_N)$ in $\mathbb{C}[z_1, \ldots, z_N]$. Let $I(\mathcal{Z}_{\mathbb{C}}(\mathbf{F}))$ be the ideal generated by all polynomials from $\mathbb{C}[z_1, \ldots, z_N]$ vanishing on $\mathcal{Z}_{\mathbb{C}}(\mathbf{F})$. It holds that $I \subset I(\mathcal{Z}_{\mathbb{C}}(\mathbf{F}))$. Using Corollary 2.15 in [47], we get that $\mathcal{Z}_{\mathbb{C}}(\mathbf{F})$ is finite if and only if the dimension of $\mathbb{C}[z_1, \ldots, z_N]/I(\mathcal{Z}_{\mathbb{C}}(\mathbf{F}))$ as a $\mathbb{C}$-vector space is finite. Since $I \subset I(\mathcal{Z}_{\mathbb{C}}(\mathbf{F}))$, there is a surjective morphism from $\mathbb{C}[z_1, \ldots, z_N]/I$ to $\mathbb{C}[z_1, \ldots, z_N]/I(\mathcal{Z}_{\mathbb{C}}(\mathbf{F}))$. Thus, it is enough to verify that the dimension of $\mathbb{C}[z_1, \ldots, z_N]/I$ as a $\mathbb{C}$-vector space is finite. Since f is a polynomial of degree $d \geq 2$, there remain in $\mathbb{C}[z_1, \ldots, z_N]/I$ the monomials $z_1^{a_1} z_2^{a_2} \ldots z_N^{a_N}$ (with $1 \leq a_i \leq (d-1)$ for all i such that $1 \leq i \leq N$), which form a set of cardinality d^N. Thus the cardinality of $\mathcal{Z}_{\mathbb{C}}(\mathbf{F})$ is bounded from above by d^N. Since this holds over the field $\mathbb{C}$, it also holds over the field $\mathbb{R}$. □

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