Proceedings of the Royal Society of Edinburgh, page 1 of 37
DOI:10.1017/prm.2023.113

Existence of renormalized solutions to fully anisotropic and inhomogeneous elliptic problems

Bartosz Budnarowski
Faculty of Mathematics, Informatics and Mechanics, University of Warsaw ul. Banacha 2, 02-097 Warsaw, Poland
(bartek.budnarowski@gmail.com)

Ying Li
Department of Mathematics, Shanghai University, Shanghai 200444, China (lyinsh@shu.edu.cn)

(Received 5 December 2022; accepted 2 October 2023)

We will present the proof of existence and uniqueness of renormalized solutions to a broad family of strongly non-linear elliptic equations with lower order terms and data of low integrability. The leading part of the operator satisfies general growth conditions settling the problem in the framework of fully anisotropic and inhomogeneous Musielak–Orlicz spaces. The setting considered in this paper generalized known results in the variable exponents, anisotropic polynomial, double phase and classical Orlicz setting.

Keywords: existence; elliptic boundary value problems; renormalized solutions; Musielak–Orlicz spaces

2020 *Mathematics Subject Classification:* 35D30; 35J60

1. Introduction

The main purpose of this paper is to establish the existence and uniqueness of solutions for a strongly non-linear elliptic equation with irregular data and very mild limitations on the growth of the operator. The leading part of the operator satisfies general growth conditions settling the problem in the framework of fully anisotropic and inhomogeneous Musielak–Orlicz spaces generated by an N-function $M : \Omega \times \mathbb{R}^n \to [0, \infty)$. Note that no growth hypothesis of doubling type is assumed on the function M. The price we pay for relaxing the condition is to assume that there is a condition balancing the behaviour of M with respect to its variable, which can ensure the density of smooth functions in the related Sobolev-type space.

Let us present our framework. Suppose that Ω is a bounded Lipschitz domain in $\mathbb{R}^n$, $n > 1$, $f : \Omega \to \mathbb{R}$, $f \in L^1(\Omega)$ and $F \in E_{M^*}(\Omega; \mathbb{R}^n)$. In this paper, we study the following problem

$$\begin{cases} -\operatorname{div}\big(\mathcal{A}(x, \nabla u) + \Phi(u)\big) + b(x, u) = f + \operatorname{div} F & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where the function $\mathcal{A} : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies the following conditions:

(A1) $\mathcal{A}$ is a Carathéodory's function (i.e. measurable with respect to the first variable and continuous with respect to the second one);

(A2) $\mathcal{A}(x, 0) = 0$ for almost every $x \in \Omega$ and there exist an N-function $M : \Omega \times \mathbb{R}^n \to [0, \infty)$ and constants $c_1^{\mathcal{A}}, c_2^{\mathcal{A}}, c_3^{\mathcal{A}}, c_4^{\mathcal{A}} > 0$ such that for all $\xi \in \mathbb{R}^n$ we have

$$\mathcal{A}(x, \xi) \cdot \xi \geqslant M(x, c_1^{\mathcal{A}}\xi)$$

and

$$c_2^{\mathcal{A}} M^*(x, c_3^{\mathcal{A}}\mathcal{A}(x, \xi)) \leqslant M(x, c_4^{\mathcal{A}}\xi);$$

(A3) For all $\xi, \eta \in \mathbb{R}^n$ and a.e. $x \in \Omega$ we have

$$(\mathcal{A}(x, \xi) - \mathcal{A}(x, \eta)) \cdot (\xi - \eta) \geqslant 0.$$

Moreover, we assume

(P) $\Phi : \mathbb{R} \to \mathbb{R}^n$ is a Lipschitz continuous function;

(b) $b : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory's function nondecreasing with respect to the second variable, and such that $b(\cdot, s) \in L^1(\Omega)$ and $b(\cdot, s)\,\mathrm{sign}\,(s) \geqslant 0$ for every $s \in \mathbb{R}$.

As it is well known when the operator $\mathcal{A}(x, \xi) = |\xi|^{p-2}\xi$ or $M(x, \xi) = |\xi|^p$, the problem is posed in the classical Sobolev setting. However, in the real world, the non-linear terms involved in the problems are often of non-standard growth. The study of differential equations with non-standard growth conditions has attracted extensive attention in recent decades. We refer to [**22**, **31**, **43**] for problems governed by conditions of (p, q)-type. Variable exponent problems were introduced in [**23**, **27**]. This paper deals with elliptic problems under conditions expressed by a generalized Orlicz function. Musielak–Orlicz spaces, which include the variable exponent, Orlicz, weighted and double-phase spaces, have been studied systematically starting from [**46**, **50**, **51**]. There have been wide research activities in the Musielak–Orlicz spaces. We refer to [**32**, **42**] for the existence of solutions in isotropic, separable and reflexive Musielak–Orlicz–Sobolev spaces. In [**30**] separable, but not reflexive Musielak–Orlicz spaces were applied. We would like to point out that non-linear elliptic boundary value problems in non-reflexive Musielak–Orlicz–Sobolev type setting were first considered by Donaldson in [**29**] and followed by Gossez [**33**, **34**]. We mention that [**20**, **21**, **40**] laid the cornerstone for studying the PDEs problem in fully anisotropic spaces. [**2**, **4**, **19**] were devoted to the study of problems in anisotropic Orlicz spaces governed by a possibly fully anisotropic modular function that is independent of the spatial variables. For the problems that are in the same time of general growth, inhomogeneous and fully anisotropic, we refer to [**16**, **17**, **24**, **39**, **41**], but none of them are concerned with the lower order terms. In particular, to comprehend the background of our problems better, we refer the readers to a monograph [**15**] and a review paper [**12**] discussing PDEs in Musielak–Orlicz spaces for details. We also mention the paper [**44**] which is a comprehensive overview of recent results concerning elliptic variational problems with non-standard growth conditions and related to different kinds of non-uniformly elliptic operators.

Our focus in this paper is to establish the existence of solutions for problem (1.1). Since we consider problems with data of low integrability, the weak solutions are not well-defined and we need to consider a generalized definition of solutions, namely renormalized solutions. The notion of renormalized solutions was first introduced by DiPerna and Lions [**28**] for the study of the Boltzmann equation. The concept was then adapted to the study of some non-linear elliptic and parabolic problems [**6**, **7**]. The existence of renormalized solutions in the variable exponents setting was considered in [**53**, **55**]. We refer to [**47**, **49**, **56**] for this issue in the non-reflexive Orlicz–Sobolev space.

There have been many articles about the renormalized solutions in Musielak–Orlicz space. Gwiazda *et al.* [**36**] proved the existence and uniqueness of renormalized solutions in the non-homogeneous and non-reflexive Musielak–Orlicz spaces for a general class of non-linear elliptic problems associated with the differential inclusion

$$\beta(x, u) - \operatorname{div}\left(\mathcal{A}(x, u) + F(u)\right) \ni f\,,$$

where $f \in L^1(\Omega)$. The growth and coercivity conditions on the monotone vector field $\mathcal{A}$ are prescribed by a generalized N-function M which is anisotropic and inhomogeneous with respect to the space variable, and Δ_2-condition was imposed on the N-function M^*. We refer to [**37**] for the corresponding parabolic problem under the same assumption on M^*. This work was then extended by Gwiazda *et al.* in [**35**] for N-function M^* not necessarily satisfying the Δ_2-condition. The authors in [**35**] proved the existence of renormalized solutions to the elliptic equation

$$-\operatorname{div} \mathcal{A}(x, \nabla u) = f \in L^1(\Omega)\,,$$

in a fully anisotropic space. In [**35**–**37**] the leading part of the operator satisfies condition

$$c_{\mathcal{A}}\left(M(x, \xi) + M^*(x, \mathcal{A}(x, \xi))\right) \leqslant \mathcal{A}(x, \xi)\xi$$

for $c_{\mathcal{A}} \in (0, 1]$, which covers more narrow family of operator than our condition $(A2)$. See [**15**, Section 3.8.2] for detailed explanation.

Inspired by the above papers, we want to extend the results obtained in [**18**, **35**, **36**]. We proved the existence of renormalized solutions for Eqn (1.1) in the setting of fully anisotropic and inhomogeneous Musielak–Orlicz spaces. Under an additional strict monotonicity assumption, uniqueness of renormalized solution is established. Many well-known results in the variable exponent, anisotropic polynomial, double phase and classical Orlicz setting are covered by our paper. We emphasize that no growth hypothesis of doubling type is assumed on the function M. Thanks to [**9**], we have the following balance condition, which gave us a sufficient condition to guarantee that the smooth function is modular dense in Musielak–Orlicz space. We shall stress that it is only applied to ensure the density of smooth functions.

Balance condition (B). Given an N-function $M : \Omega \times \mathbb{R}^n \to [0, \infty)$ suppose there exists a constant $C_M > 1$ such that for every ball $B \subset \Omega$ with $|B| \leqslant 1$, every $x \in B$, and for all $\xi \in \mathbb{R}^n$ such that $|\xi| > 1$ and $M(x, C_M\xi) \in [1, \frac{1}{|B|}]$ there holds $\sup_{y\in B} M(y, \xi) \leqslant M(x, C_M\xi)$.

Note that in the isotropic and doubling regime, this condition is known to be sufficient to the boundedness of the maximal operator. Moreover, when $d = 1$ they are equivalent [**38**]. Condition (B) is essentially less restrictive than the isotropic one from [**1**] or the anisotropic ones used in [**15**, **35**]. Following [**9**, **38**], we give examples of N-functions satisfying the above balance condition. See also [**10**] for more general condition than (B).

Example 1.1. The following N-functions fit into our setting.

- Variable exponent case: $M(x, \xi) = |\xi|^{p(x)}$, where $p(x) : \Omega \to [p^-, p^+]$ is log-Hölder continuous and $1 < p^- \leqslant p(\cdot) \leqslant p^+ \leqslant \infty$; see the proof of [**38**, Proposition 7.1.2].

- Double phase case: $M(x, \xi) = |\xi|^p + a(x)|\xi|^q$, with $1 < p \leqslant q < \infty$, $0 \leqslant a \in C^{0,\alpha}(\Omega)$, $\alpha \in (0, 1]$, $q/p \leqslant 1 + \alpha/n$; see the proof of [**38**, Proposition 7.2.2].

- Anisotropic variable case: $M(x, \xi) = \sum_{i=1}^n |\xi_i|^{p_i(x)}$, where $p_i(x) : \Omega \to [p_i^-, p_i^+]$ are log-Hölder continuous and $1 < p_i^- \leqslant p_i(\cdot) \leqslant p_i^+ \leqslant \infty$; see [**9**, Subsection 4.4.].

- Anisotropic double phase case: $M(x, \xi) = \sum_{i=1}^n (|\xi_i|^{p_i} + a_i(x)|\xi_i|^{q_i})$, where $1 < p_i \leqslant q_i < \infty$, $0 \leqslant a_i \in C^{0,\alpha_i}(\Omega)$, $\alpha_i \in (0, 1]$, and $q_i/p_i \leqslant 1 + \alpha_i/n$; as well as anisotropic multi-phase case (also with Orlicz phases), see [**9**, Subsection 4.4.].

Taking into account [**19**] and [**9**, Section 4] one can provide an explicit condition that implies (B) even in the case when the anisotropic function $M(x, \xi)$ does not admit a so-called orthotropic decomposition $\sum_{i=1}^d M_i(x, \xi_i)$ even after an affine change of variables.

Before we give the definition of renormalized solution to (1.1). We shall introduce the truncation $T_k(s)$ as follows

$$T_k(s)(x) = \begin{cases} s & |s| \leqslant k, \\ k\dfrac{s}{|s|} & |s| \geqslant k. \end{cases} \tag{1.2}$$

Note that as a consequence of Lemma 2.1 of [**5**], for every measurable function u on Ω such that $T_k(u) \in V_0^1 L_M$ for every $k > 0$, there exists a unique measurable function $Z_u : \Omega \to \mathbb{R}^N$ such that

$$\nabla T_k(u) = \chi_{\{|u|<k\}} Z_u \text{ for almost every } x \in \Omega \text{ and for every } k > 0,$$

where χ_E denotes the characteristic function of a measurable set E. We will understand ∇u as a pointwise limit of $\nabla T_k(u)$ as $k \to \infty$.

Definition 1.2. *We call a function u a renormalized solution to* (1.1)*, when it satisfies the following conditions:*

(R1) $u : \Omega \to \mathbb{R}$ *is measurable and for each* $k > 0$

$$T_k(u) \in V_0^1 L_M(\Omega) \cap L^\infty(\Omega), \quad \mathcal{A}(x, \nabla T_k(u)) \in L_{M^*}(\Omega; \mathbb{R}^n),$$

where

$$V_0^1 L_M(\Omega) := \{\varphi \in W_0^{1,1}(\Omega) : \nabla\varphi \in L_M(\Omega; \mathbb{R}^n)\}.$$

(R2) *For every* $h \in C_c^1(\mathbb{R})$ *and all* $\varphi \in V_0^1 L_M(\Omega) \cap L^\infty(\Omega)$ *we have*

$$\int_\Omega \mathcal{A}(x, \nabla u) \cdot \nabla(h(u)\varphi) + \Phi(u) \cdot \nabla(h(u)\varphi) + b(x, u)h(u)\varphi \,\mathrm{d}x$$
$$= \int_\Omega f h(u)\varphi + F \cdot \nabla(h(u)\varphi) \,\mathrm{d}x.$$

(R3) $\int_{\{l<|u|<l+1\}} \mathcal{A}(x, \nabla u) \cdot \nabla u \,\mathrm{d}x \to 0$ *as* $l \to \infty$.

Our main result reads as follows.

Theorem 1.3. *Suppose $f \in L^1(\Omega)$, $F \in E_{M^*}(\Omega; \mathbb{R}^n)$, an N-function M is regular enough so that $C_c^\infty(\Omega)$ is dense in $V_0^1 L_M(\Omega)$ in the modular topology. Function $\mathcal{A}$ satisfies assumptions (A1), (A2) and (A3), Φ satisfies (P), and b satisfies (b). Then there exists at least one renormalized solution to the problem*

$$\begin{cases} -\operatorname{div}\big(\mathcal{A}(x, \nabla u) + \Phi(u)\big) + b(x, u) = f + \operatorname{div} F & \text{in} \quad \Omega, \\ u(x) = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

Namely, there exists u, which satisfies (R1)–(R3).

Proposition 1.4. *Under the assumptions of theorem* 1.3*, if we assume that $s \to b(\cdot, s)$ is strictly increasing, then the renormalized solution is unique.*

We briefly introduce our approach to the proof of our main results. Our growth conditions put the problem in an inhomogeneous and fully anisotropic setting. We address the challenges that come from the lacking of the growth condition and the presence of lower order terms. The main difficulty lies in that there are no conditions of doubling-type assumed for function $M(x, \xi)$ as it was done in [**13**, **36**]. It complicates the understanding of the dual pairing since L_M is not dual of L_{M^*} in general. We consider $(A2)$, which is a more general growth condition than those employed in [**15**, **35**, **36**]. This essentially affects the derivation of *a priori* estimates. The classical results are not applicable due to the generality of the situation considered, such as Sobolev embeddings or Rellich–Kondrachov compact embeddings. There is no good embedding of fully anisotropic Musielak–Orlicz–Sobolev spaces into Musielak–Orlicz spaces. The appearance of lower-order terms complicates the analysis of the problem. This is particularly well visible in the identification of some limits in our approximate procedure (step 5 of the proof of theorem 1.1), as well as in the argumentation that the limit of the approximation shares properties of

https://doi.org/10.1017/prm.2023.113 Published online by Cambridge University Press

renormalized solutions (step 6). There are just a few results that deal with the anisotropic problems with lower-order terms. We can only refer to [25, 36], but they do not cover the generality of the problem. Moreover, unlike the operator considered in [3, 54–56], we do not need the operator of the problem (1.1) to be strictly monotone. Also, the low integrability of the right-hand terms leads to significant difficulties in convergence studies. The set of smooth functions is not dense in the norm topology in the general Orlicz–Sobolev spaces, so we need to impose a balance condition, which can ensure the density of smooth functions in the related Sobolev-type space.

This paper is organized as follows. In § 2, we state some basic results that will be used later. We will prove the main results in § 3. Uniqueness of the renormalized solution will be proved in § 4.

2. Preliminary lemmas

In this section, we introduce some fundamental definitions and auxiliary results. By Ω we always mean a bounded domain of $\mathbb{R}^n$ with Lipschitz regular boundary. If not specified, a constant C is a positive constant, possibly changing line by line. By $C_c^\infty(\Omega)$ we mean the set of compactly supported smooth functions over Ω. We begin with N-functions and the Musielak–Orlicz space setting.

Definition 2.1. *A function $M(x,\xi):\Omega\times\mathbb{R}^n\to\mathbb{R}$ is called an N-function if*

- *M is a Carathéodory function;*
- *$M(x,0)=0$ and $\xi\mapsto M(x,\xi)$ is a convex function with respect to ξ for a.a. $x\in\Omega$;*
- *$M(x,\xi)=M(x,-\xi)$ for a.a. $x\in\Omega$ and all $\xi\in\mathbb{R}^n$;*
- *there exist two convex functions $m_1, m_2:[0,\infty)\to[0,\infty)$ such that*

$$\lim_{s\to0^+}\frac{m_1(s)}{s}=0=\lim_{s\to0^+}\frac{m_2(s)}{s}\quad\textit{and}\quad\lim_{s\to\infty}\frac{m_1(s)}{s}=\infty=\lim_{s\to\infty}\frac{m_2(s)}{s},$$

and for a.a. $x\in\Omega$

$$m_1(|\xi|)\leqslant M(x,\xi)\leqslant m_2(|\xi|).$$

For an N-function we define the general Musielak–Orlicz class $\mathcal{L}_M(\Omega;\mathbb{R}^n)$ as the set of all measurable functions $\xi(x):\Omega\to\mathbb{R}^n$ such that

$$\int_\Omega M(x,\xi(x))\,\mathrm{d}x<\infty.$$

The Musielak–Orlicz space $L_M(\Omega;\mathbb{R}^n)$ is the smallest linear hull of $\mathcal{L}_M(\Omega;\mathbb{R}^n)$ equipped with the Luxemburg norm

$$\|\xi\|_{L_M(\Omega)}=\inf\left\{\lambda>0:\int_\Omega M\left(x,\frac{\xi(x)}{\lambda}\right)\mathrm{d}x\leqslant1\right\}.$$

The space $E_M(\Omega;\mathbb{R}^n)$ is the closure in L_M-norm of the set of bounded functions. Equivalently, $L_M(\Omega;\mathbb{R}^n)$ and $E_M(\Omega;\mathbb{R}^n)$ are defined as sets of functions $\xi:\Omega\to\mathbb{R}^n$ satisfying

$$\int_\Omega M(x,\lambda\xi(x))\,\mathrm{d}x<\infty$$

for some $\lambda\in\mathbb{R}$ and for every $\lambda\in\mathbb{R}$, respectively [**15**, Lemma 3.1.8]. We also note that $(E_M(\Omega;\mathbb{R}^n))^*=L_{M^*}(\Omega;\mathbb{R}^n)$ and $(E_{M^*}(\Omega;\mathbb{R}^n))^*=L_M(\Omega;\mathbb{R}^n)$ [**15**, Theorem 3.5.3] but no other duality relations are expected.

The complementary function to M is

$$M^*(x,\eta):=\sup_{\xi\in\mathbb{R}^n}(\xi\cdot\eta-M(x,\xi)).$$

If M is an N-function and M^* is the complementary function to M, then the following Fenchel–Young inequality is satisfied

$$|\xi\cdot\eta|\leqslant M(x,\xi)+M^*(x,\eta)\quad\text{for all }\ \xi,\eta\in\mathbb{R}^n\text{ and a.e. }x\in\Omega.$$

Moreover, if M is an N-function and M^* its complementary, then the generalized Hölder inequality holds, e.g.

$$\left|\int_\Omega\xi\cdot\eta\,\mathrm{d}x\right|\leqslant 2\|\xi\|_{L_M}\|\eta\|_{L_{M^*}}\qquad\text{for all }\xi\in L_M(\Omega;\mathbb{R}^n)\text{ and }\eta\in L_{M^*}(\Omega;\mathbb{R}^n).$$

We say that a sequence $\{\xi_n\}_{n=1}^\infty\subset L_M(\Omega;\mathbb{R}^n)$ converges modularly to ξ in $L_M(\Omega;\mathbb{R}^n)$, if there exists $\lambda>0$ such that

$$\int_\Omega M\left(x,\frac{\xi_n-\xi}{\lambda}\right)\mathrm{d}x\to 0\text{ as }n\to\infty.$$

For the notion of this convergence, we write $\xi_n\xrightarrow{M}\xi$.

Then, we shall give some preliminary lemmas related to N-functions and Musielak–Orlicz spaces.

Proposition 2.2 [**9**, Theorem 1]. *Assume that Ω is a bounded Lipschitz domain and M is an N-function which satisfies the balance condition (B). Then, for every $u\in V_0^1L_M(\Omega)$ there exists a sequence $\{u_\delta\}_\delta\subset C_c^\infty(\Omega)$ such that*

$$u_\delta\to u\ \textit{in}\ L^1(\Omega)\ \textit{and}\ \nabla u_\delta\xrightarrow{M}\nabla u\ \textit{in}\ L_M(\Omega;\mathbb{R}^n)$$

Furthermore, there exists a constant $c=c(\Omega)$, such that $||u_\delta||_{L^\infty(\Omega)}\leqslant c||u||_{L^\infty(\Omega)}$.

Lemma 2.3 de la Vallée Poussin theorem [**15**, Lemma 3.4.2]. *Suppose M is an N-function and let $\{\xi_n\}_{n\in\mathbb{N}}$ be a sequence of measurable functions $\xi_n:\Omega\to\mathbb{R}^n$ satisfying*

$$\sup_{n\in\mathbb{N}}\int_\Omega M(x,\xi_n(x))\,\mathrm{d}x<\infty.$$

Then the sequence $\{\xi_n\}_{n\in\mathbb{N}}$ is uniformly integrable in $L^1(\Omega)$.

https://doi.org/10.1017/prm.2023.113 Published online by Cambridge University Press

LEMMA 2.4 [**15**, Lemma 3.1.14]. *Let M be an N-function.*

(1) *If $\xi \in L_M(\Omega;\mathbb{R}^n)$ and $\|\xi\|_{L_M} \leqslant 1$, then $\int_\Omega M(x,\xi(x))\,\mathrm{d}x \leqslant \|\xi\|_{L_M}$.*

(2) *If $\xi \in L_M(\Omega;\mathbb{R}^n)$ and $\|\xi\|_{L_M} > 1$, then $\int_\Omega M(x,\xi(x))\,\mathrm{d}x \geqslant \|\xi\|_{L_M}$.*

LEMMA 2.5 [**15**, Lemma 3.8.2]. *Suppose M is an N-function and $\mathcal{A}:\Omega\times\mathbb{R}^n\to\mathbb{R}^n$ satisfies (A1), (A2), (A3) and suppose $\|\mathcal{A}(\cdot,\xi)\cdot\xi\|_{L^1(\Omega)} \leqslant \tilde{c}$. Then there exists a constant $C>0$ depending only on the parameters from (A1), (A2) and $\tilde{c}$ such that $\|\mathcal{A}(\cdot,\xi)\|_{L_{M^*}} < C$.*

Next, we point out that the existence of weak solutions to the following problem follows directly from [**18**, Theorem 1.1].

PROPOSITION 2.6. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Suppose that an N-function M is regular enough so that $C_c^\infty(\Omega)$ is dense in $V_0^1L_M(\Omega)$ in the modular topology. Assume further that $g \in L^\infty(\Omega)$, $F \in E_{M^*}(\Omega;\mathbb{R}^n)$, function $\mathcal{A}$ satisfies assumptions (A1), (A2) and (A3), Φ is a bounded and continuous function, and b satisfies (b). Then there exists a weak solution to the problem*

$$\begin{cases} -\operatorname{div}\big(\mathcal{A}(x,\nabla u)+\Phi(u)\big)+b(x,u)=g+\operatorname{div}F & \text{in} \quad \Omega,\\ u(x)=0 & \text{on} \quad \partial\Omega, \end{cases}$$

Namely, there exists a function $u \in V_0^1L_M(\Omega)$ satisfying

$$\int_\Omega \mathcal{A}(x,\nabla u)\cdot\nabla\phi+\Phi(u)\cdot\nabla\phi+b(x,u)\phi\,\mathrm{d}x=\int_\Omega g\phi\,\mathrm{d}x+\int_\Omega F\cdot\nabla\phi\,\mathrm{d}x$$

for all $\phi \in V_0^1L_M(\Omega)\cap L^\infty(\Omega)$.

In fact, for each $g \in L^\infty(\Omega)$, we know that there exists $H:\Omega\to\mathbb{R}^n$, such that $g=\operatorname{div}H$ and $H \in E_{M^*}(\Omega;\mathbb{R}^n)$. The fact one can take $H \in E_{M^*}(\Omega;\mathbb{R}^n)$ is a consequence of properties of Bogovski operator. This is explained in [**15**, Remark 4.1.7] with the use of [**52**, Lemma II.2.1.1].

LEMMA 2.7 [**15**, Theorem 4.1.1]. *Suppose $\boldsymbol{A}:\Omega\times\mathbb{R}^n\to\mathbb{R}^n$ satisfies condition (A1)–(A2) with an N-function $M:\Omega\times\mathbb{R}^n\to[0,\infty)$. Moreover, assume that there exist*

$$\boldsymbol{\alpha} \in L_{M^*}(\Omega;\mathbb{R}^n) \text{ and } \boldsymbol{\xi} \in L_M(\Omega;\mathbb{R}^n)$$

such that

$$\int_\Omega \big(\boldsymbol{\alpha}-\boldsymbol{A}(x,\boldsymbol{\eta})\big)\cdot(\boldsymbol{\xi}-\boldsymbol{\eta})\,\mathrm{d}x \geqslant 0 \text{ for all } \boldsymbol{\eta}\in\mathbb{R}^n.$$

Then

$$\boldsymbol{A}(x,\boldsymbol{\xi})=\boldsymbol{\alpha} \text{ a.e. in } \Omega.$$

https://doi.org/10.1017/prm.2023.113 Published online by Cambridge University Press

LEMMA 2.8 [**15**, Lemma 8.22]. *Suppose* $z_s \overset{s\to\infty}{\rightharpoonup} z$ *in* $L^1(\Omega)$ *and* $w_s, w \in L^\infty(\Omega)$. *Assume further that there exists a constant* $C > 0$ *such that* $\sup_{s\in\mathbb{N}} ||w_s||_\infty < C$ *and* $w_s \xrightarrow[s\to\infty]{a.e.} w$. *Then*

$$\lim_{s\to\infty} \int_\Omega w_s z_s \,\mathrm{d}x = \int_\Omega wz \,\mathrm{d}x.$$

Young measures are now a standard tool for non-linear analysis. We will need the following version of the generalized fundamental theorem on Young measures from [**15**], where by $\mathcal{M}(\mathbb{R}^n)$ we denote the space of bounded Radon measures. A sequence $\{z_j\}_{j\in\mathbb{N}}$ of measurable function $z_j : \Omega \to \mathbb{R}^n$ is said to satisfy the tightness condition if

$$\lim_{R\to\infty} \sup_{j\in\mathbb{N}} |\{x : |z_j(x)| \geqslant R\}| = 0.$$

LEMMA 2.9 **Fundamental theorem for Young measures** [**15**, Theorem 8.41]. *Let* $\Omega \subset \mathbb{R}^n$ *and* $z_j : \Omega \to \mathbb{R}^n$ *be a sequence of measurable functions. Then there exists a subsequence* $\{z_j\}$ *and a family of weakly-∗ measurable maps* $\nu_x : \Omega \to \mathcal{M}(\mathbb{R}^n)$*, such that:*

(1) $\nu_x \geqslant 0$, $\|\nu_x\|_{\mathcal{M}(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \mathrm{d}\nu_x \leqslant 1$ *for a.e.* $x \in \Omega$.

(2) *For every* $f \in C_0(\mathbb{R}^n)$*, we have* $f(z_j) \overset{*}{\rightharpoonup} \bar{f}$ *weakly-∗ in* $L^\infty(\Omega)$. *Moreover,*

$$\bar{f} = \int_{\mathbb{R}^n} f(\lambda) \,\mathrm{d}\nu_x(\lambda).$$

(3) *Let* $K \subset \mathbb{R}^n$ *be compact and* $\mathrm{dist}(z_j, K) \to 0$ *in measure, then supp* $v_x \subset K$.

(4) $\|\nu_x\|_{\mathcal{M}(\mathbb{R}^n)} = 1$ *for a.e.* $x \in \Omega$ *if and only if the tightness condition is satisfied.*

(5) *If the tightness condition is satisfied,* $E \subset \Omega$ *is measurable,* $f \in C(\mathbb{R}^n)$*, and* $\{f(z_j)\}$ *is relatively weakly compact in* $L^1(E)$*, then*

$$f(z_j) \rightharpoonup \bar{f} \quad in \quad L^1(E) \quad and \quad \bar{f} = \int_{\mathbb{R}^n} f(\lambda) \,\mathrm{d}\nu_x(\lambda).$$

The family of maps $\nu_x : \Omega \to \mathcal{M}(\mathbb{R}^n)$ *is called the Young measure generated by* $\{z_j\}$.

LEMMA 2.10 [**45**, Corollary 3.3]. $z^j : \Omega \to \mathbb{R}^n$ *generates the Young measure* v, $B : \Omega \times \mathbb{R}^n \to \mathbb{R}^+$ *is a Carathéodory function. Then*

$$\liminf_{j\to\infty} \int_\Omega B(x, z^j(x)) \,\mathrm{d}x \geqslant \int_\Omega \int_{\mathbb{R}^n} B(x, \lambda) \mathrm{d}v_x(\lambda) \,\mathrm{d}x.$$

DEFINITION 2.11 **Biting convergence** [**15**, Definition 8.36]. *Let* $f_j, f \in L^1(\Omega)$ *for every* $j \in \mathbb{N}$. *We say that a sequence* $\{f_j\}_{j\in\mathbb{N}}$ *converges in the sense of biting to* f *in* $L^1(\Omega)$ *(and denote it by* $f_j \xrightarrow{b} f$*), if there exists a sequence* E_j *of measurable subsets of* Ω *such that* $\lim_{j\to\infty} |E_j| = 0$ *and for every* j *we have* $f_j \rightharpoonup f$ *in* $L^1(\Omega \setminus E_j)$.

https://doi.org/10.1017/prm.2023.113 Published online by Cambridge University Press

Lemma 2.12 **Chacon's biting lemma** [48, Lemma 6.6]. *Let $\Omega \in \mathbb{R}^n$ be a measurable set and let the sequence $\{f_j\}_j^\infty \subset L^1(\Omega)$ be bounded in $L^1(\Omega)$. There exists a subsequence of indices, still denoted by j, and a function $f \in L^1(\Omega)$ such that $f_j \xrightarrow{b} f$.*

Lemma 2.13 [48, Lemma 6.9]. *Let $f_j \in L^1(\Omega)$ for every $j \in \mathbb{N}$, $0 \leqslant f_j(x)$ for a.e. $x \in \Omega$. Moreover, suppose*

$$f_j \xrightarrow{b} f \quad \textit{and} \quad \limsup_{j\to\infty} \int_\Omega f_j \,\mathrm{d}x \leqslant \int_\Omega f \,\mathrm{d}x.$$

Then

$$f_j \rightharpoonup f \quad \textit{weakly in} \quad L^1(\Omega) \quad \textit{for} \quad j \to \infty.$$

3. Existence of renormalized solutions—main proof

Now, we are in the position to prove our main result. The whole proof is divided into 6 steps. We begin with the existence of a solution to a problem with truncated data, then we show *a priori* estimates and the energy control condition for solutions to the problem. After that we focus on the most challenging part—passing to the limit with the level of truncation. In the last step, we show that the function u that we obtained as a weak limit, is in fact a renormalized solution. This is the place where Young measures appear.

Proof of theorem 1.3. **Step 1. Problems with truncated data.**
The existence of a weak solution to the problem

$$\begin{cases} -\operatorname{div}\big(\mathcal{A}(x,\nabla u) + \Phi_s(u)\big) + b(x,u) = T_s(f) + \operatorname{div} F & \text{in} \quad \Omega, \\ u(x) = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

for every $s > 0$ is a consequence of proposition 2.6 with $g = T_s(f)$, $\Phi_s(u) = \Phi(T_s(u))$. Namely, there exists a function $u_s \in V_0^1 L_M(\Omega)$ satisfying

$$\int_\Omega \mathcal{A}(x,\nabla u_s)\cdot\nabla\phi + \Phi_s(u_s)\cdot\nabla\phi + b(x,u_s)\phi \,\mathrm{d}x = \int_\Omega T_s(f)\phi\,\mathrm{d}x + \int_\Omega F\cdot\nabla\phi\,\mathrm{d}x \tag{3.1}$$

for all $\phi \in V_0^1 L_M(\Omega) \cap L^\infty(\Omega)$.

Step 2. *A priori* estimates
We test the function (3.1) by $T_k(u_s)$ to get

$$\int_\Omega \mathcal{A}\big(x,\nabla T_k(u_s)\big)\cdot\nabla T_k(u_s)\,\mathrm{d}x + \int_\Omega \Phi(T_s(u_s))\cdot\nabla T_k(u_s)\,\mathrm{d}x + \int_\Omega b(x,u_s)T_k(u_s)\,\mathrm{d}x$$
$$= \int_\Omega T_s(f)T_k(u_s) + F\nabla T_k(u_s)\,\mathrm{d}x.$$

Using condition (A2) we get an estimate

$$\frac{1}{2}\int_\Omega M\big(x, c_1^{\mathcal{A}}\nabla T_k(u_s)\big)\,\mathrm{d}x \leqslant \frac{1}{2}\int_\Omega A\big(x,\nabla T_k(u_s)\big)\cdot\nabla T_k(u_s)\,\mathrm{d}x.$$

https://doi.org/10.1017/prm.2023.113 Published online by Cambridge University Press

Since Φ is continuous, we can apply the chain rule theorem for Sobolev functions. We notice that there exists a function $G : \mathbb{R} \to \mathbb{R}^n$ such that $G(0) = 0$ and we have $\operatorname{div} G(T_k(u_s)) = \Phi(T_k(u_s)) \cdot \nabla T_k(u_s)$. Therefore, the Gauss–Green theorem yields:

$$\int_\Omega \Phi(T_k(u_s)) \cdot \nabla T_k(u_s)\,\mathrm{d}x = 0.$$

Moreover, the Fenchel–Young inequality and the definition of N functions allow us to infer that

$$\begin{aligned}\int_\Omega \frac{4}{c_1^{\mathcal{A}}} F \cdot \frac{c_1^{\mathcal{A}}}{4} \nabla T_k(u_s)\mathrm{d}x &\leqslant \int_\Omega M\left(x, \frac{c_1^{\mathcal{A}}}{4}\nabla T_k(u_s)\right)\mathrm{d}x + \int_\Omega M^*\left(x, \frac{4}{c_1^{\mathcal{A}}}F\right)\mathrm{d}x \\ &\leqslant \frac{1}{4}\int_\Omega M\big(x, c_1^{\mathcal{A}}\nabla T_k(u_s)\big)\mathrm{d}x + \int_\Omega M^*\left(x, \frac{4}{c_1^{\mathcal{A}}}F\right)\mathrm{d}x.\end{aligned}$$

And thus, according to $M(x, \xi) \geqslant 0$ for almost every $x \in \Omega$ and every $\xi \in \mathbb{R}^n$, $b(x, u_s)T_k(u_s) \geqslant 0$ as b satisfies condition (b), we have

$$\begin{aligned}&\frac{1}{4}\int_\Omega M\big(x, c_1^{\mathcal{A}}\nabla T_k(u_s)\big)\,\mathrm{d}x + \frac{1}{2}\int_\Omega \mathcal{A}\big(x, \nabla T_k(u_s)\big)\cdot\nabla T_k(u_s)\,\mathrm{d}x \\ &\quad\leqslant \int_\Omega M^*\left(x, \frac{4}{c_1^{\mathcal{A}}}F\right) + k\|f\|_{L^1(\Omega)}.\end{aligned}$$

Since $F \in E_{M^*}(\Omega)$ by assumption, the right-hand side of the latter inequality is finite and we infer that

$$\int_\Omega \mathcal{A}\big(x, \nabla T_k(u_s)\big)\cdot\nabla T_k(u_s)\,\mathrm{d}x \leqslant C$$

and

$$\int_\Omega M(x, c_1^{\mathcal{A}}\nabla T_k(u_s))\,\mathrm{d}x \leqslant C.$$

Furthermore, by lemmas 2.4 and 2.5, we have

$$\|\nabla T_k(u_s)\|_{L_M} \leqslant \frac{1}{c_1^{\mathcal{A}}}\left(\int_\Omega M\big(x, c_1^{\mathcal{A}}\nabla T_k(u_s)\big)\,\mathrm{d}x + 1\right) \leqslant C \tag{3.2}$$

and

$$\|\mathcal{A}(\cdot, \nabla T_k(u_s))\|_{L_{M^*}} \leqslant C, \tag{3.3}$$

where C is independent of s. □

Step 3. Energy control.

Now we will show that for every weak solution u_s to (3.1) there exists a $\gamma : [0, \infty) \to [0, \infty)$ independent of s and l such that $\lim_{t\to 0}\gamma(t) = 0$ and for every

https://doi.org/10.1017/prm.2023.113 Published online by Cambridge University Press

$l > 0$

$$\int_{\{l<|u_s|<l+1\}} \mathcal{A}(x, \nabla u_s) \cdot \nabla u_s \, \mathrm{d}x \leqslant \gamma\left(\frac{l}{m_1(c_1 l)}\right) \tag{3.4}$$

for some $c_1 = c_1(\Omega) > 0$, where m_1 is the minorant of M in the definition of an N-function.

Considering properties of truncations, we infer

$$\begin{aligned}\int_{\{l<|u_s|<l+1\}} \mathcal{A}(x, \nabla u_s) \cdot \nabla u_s \, \mathrm{d}x &= \int_{\{l<|u_s|<l+1\}} \mathcal{A}\big(x, \nabla T_{l+1}(u_s)\big) \cdot \nabla T_{l+1}(u_s) \, \mathrm{d}x \\ &= \int_\Omega \mathcal{A}(x, \nabla u_s) \cdot \nabla\big(T_{l+1}(u_s) - T_l(u_s)\big) \, \mathrm{d}x.\end{aligned} \tag{3.5}$$

Testing (3.1) by $(T_{l+1}(u_s) - T_l(u_s))$, we have

$$\begin{aligned}&\int_{\{l<|u_s|<l+1\}} \mathcal{A}\big(x, \nabla T_{l+1}(u_s)\big) \cdot \nabla T_{l+1}(u_s) \, \mathrm{d}x + \int_{\{l<|u_s|<l+1\}} \Phi_s(u_s) \cdot \nabla T_{l+1}(u_s) \, \mathrm{d}x \\ &\quad + \int_{\{l\leqslant|u_s|\}} b(x, u_s)\big(T_{l+1}(u_s) - T_l(u_s)\big) \, \mathrm{d}x \\ &= \int_\Omega \mathcal{A}(x, \nabla u_s) \cdot \nabla\big(T_{l+1}(u_s) - T_l(u_s)\big) \, \mathrm{d}x + \int_\Omega \Phi_s(u_s) \cdot \nabla\big(T_{l+1}(u_s) - T_l(u_s)\big) \, \mathrm{d}x \\ &\quad + \int_\Omega b(x, u_s)\big(T_{l+1}(u_s) - T_l(u_s)\big) \, \mathrm{d}x \\ &= \int_\Omega T_s(f)\big(T_{l+1}(u_s) - T_l(u_s)\big) \, \mathrm{d}x + \int_\Omega F\nabla\big(T_{l+1}(u_s) - T_l(u_s)\big) \, \mathrm{d}x \\ &\leqslant \int_{\{|u_s|\geqslant l\}} |f| \, \mathrm{d}x + \int_{\{l\leqslant|u_s|<l+1\}} F\nabla T_{l+1}(u_s) \, \mathrm{d}x.\end{aligned}$$

Since $\int_\Omega \Phi_s(u_s) \cdot \nabla(T_{l+1}(u_s) - T_l(u_s)) \, \mathrm{d}x = 0$ and the term involving function b can be dropped as it is non-negative, we get

$$\begin{aligned}\int_{\{l<|u_s|<l+1\}} \mathcal{A}(x, \nabla T_{l+1}(u_s)) \cdot \nabla T_{l+1}(u_S) \, \mathrm{d}x &\leqslant \int_{\{l\leqslant|u_s|\}} |f| \, \mathrm{d}x \\ &\quad + \int_{\{l\leqslant|u_s|<l+1\}} F\nabla T_{l+1}(u_s) \, \mathrm{d}x.\end{aligned}$$

Moreover,

$$\begin{aligned}\int_{\{l\leqslant|u_s|<l+1\}} F\nabla T_{l+1}(u_s) \, \mathrm{d}x &\leqslant \frac{1}{4} \int_{\{l\leqslant|u_s|<l+1\}} M\big(x, c_1^{\mathcal{A}} \nabla T_{l+1}(u_s)\big) \mathrm{d}x \\ &\quad + \int_{\{l\leqslant|u_s|<l+1\}} M^*\left(x, \frac{4}{c_1^{\mathcal{A}}} F\right) \, \mathrm{d}x.\end{aligned}$$

https://doi.org/10.1017/prm.2023.113 Published online by Cambridge University Press

Therefore, we have

$$\begin{aligned}\frac{1}{4}\int_{\{l<|u_s|<l+1\}} M\big(x, c_1^{\mathcal{A}}\nabla T_{l+1}(u_s)\big)\mathrm{d}x + \frac{1}{2}\int_{\{l<|u_s|<l+1\}} \mathcal{A}(x,\nabla u_s)\cdot\nabla u_s\,\mathrm{d}x \\ \leqslant \int_{\{l\leqslant|u_s|\}} |f|\,\mathrm{d}x + \int_{\{l\leqslant|u_s|\}} M^*\left(x, \frac{4}{c_1^{\mathcal{A}}}F\right)\mathrm{d}x.\end{aligned} \tag{3.6}$$

We want to estimate the right-hand side of (3.6). To do so, we must find some control over the measure of the set $\{|u_s|\geqslant l\}$, which is the domain of integration. Note that for m_1 we obtain

$$|\{|u_s|\geqslant l\}| = |\{|T_l(u_s)| = l\}| = |\{|T_l(u_s)|\geqslant l\}| = \big|\big\{m_1\big(c_1|T_l(u_s)|\big)\geqslant m_1(c_1 l)\big\}\big|.$$

Applying the Chebyshev inequality [8, Theorem 2.5.3], the modular Poincaré inequality [15, Theorem 9.3] involving m_1 and using the fact it is a convex minorant of M, we arrive at

$$\begin{aligned}|\{|u_s|\geqslant l\}| &\leqslant \int_\Omega \frac{m_1\big(c_1|T_l(u_s)|\big)}{m_1(c_1 l)}\,\mathrm{d}x \\ &\leqslant \frac{c_2}{m_1(c_1 l)}\int_\Omega m_1\big(\big|c_1^{\mathcal{A}}\nabla T_l(u_s)\big|\big)\,\mathrm{d}x \\ &\leqslant \frac{c_2}{m_1(c_1 l)}\int_\Omega M\big(x, c_1^{\mathcal{A}}\nabla T_l(u_s)\big)\,\mathrm{d}x \\ &\leqslant C\frac{l}{m_1(c_1 l)}.\end{aligned} \tag{3.7}$$

Since m_1 is an N-function, it is superlinear at infinity. Hence the right-hand side of the inequality above vanishes when $l\to\infty$. As a result, there exists $\gamma:[0,\infty)\to[0,\infty)$ independent of s and l, such that $\lim_{t\to 0}\gamma(t)=0$ and we have

$$\int_A M^*\left(x,\frac{4}{c_1^{\mathcal{A}}}F\right) + |f|\,\mathrm{d}x \leqslant \frac{1}{2}\gamma(|A|).$$

Thanks to (3.7), we get

$$\int_{\{l\leqslant|u_s|\}} M^*\left(x,\frac{4}{c_1^{\mathcal{A}}}F\right) + |f|\,\mathrm{d}x \leqslant \frac{1}{2}\gamma\left(\frac{l}{m_1(c_1 l)}\right). \tag{3.8}$$

Combining this with (3.6) we arrive at the claim, which is

$$\int_{\{l<|u_s|<l+1\}} \mathcal{A}(x,\nabla u_s)\cdot\nabla u_s\,\mathrm{d}x \leqslant \gamma\left(\frac{l}{m_1(c_1 l)}\right).$$

https://doi.org/10.1017/prm.2023.113 Published online by Cambridge University Press

Step 4. Convergence of trucations.

In this part of the proof, we would like to show that there exists a subsequence of $\{u_s\}_{s>0}$ which has a limit $u:\Omega\to\mathbb{R}$ in the sense that

$$u_s \xrightarrow[s\to\infty]{} u \text{ a.e. in } \Omega \tag{3.9}$$

such that $T_k(u)\in V_0^1L_M(\Omega)$ for every $k>0$ and also

$$|\{|u|>l\}| \xrightarrow[l\to\infty]{} 0. \tag{3.10}$$

For every $k\in\mathbb{N}$, passing with $s\to\infty$ we have

$$\begin{aligned} T_k(u_s) \to T_k(u)&\text{strongly in } L^1(\Omega),\\ \nabla T_k(u_s) \rightharpoonup \nabla T_k(u)&\text{weakly in } L^1(\Omega;\mathbb{R}^n),\\ \nabla T_k(u_s) \xrightharpoonup{*} \nabla T_k(u)&\text{weakly}-* \text{ in } L_M(\Omega;\mathbb{R}^n),\\ \mathcal{A}(x,\nabla T_k(u_s)) \xrightharpoonup{*} \mathcal{A}_k&\text{weakly-}* \text{ in } L_{M^*}(\Omega;\mathbb{R}^n) \end{aligned} \tag{3.11}$$

for some $\mathcal{A}_k\in L_{M^*}(\Omega;\mathbb{R}^n)$.

Fix $k\in\mathbb{N}$. We have already proven the following *a priori* estimate (3.2), namely

$$\|\nabla T_k(u_s)\|_{L_M}\leqslant C.$$

Using the Banach–Alaoglu theorem [**11**, Corollary 3.30] we infer that the sequence $\{\nabla T_k(u_s)\}_{s>0}$ is weakly-$*$ compact in $L_M(\Omega;\mathbb{R}^n)$. The fact that M is an N-function together with lemma 2.3 imply that $\{\nabla T_k(u_s)\}_{s>0}$ is uniformly integrable $L^1(\Omega;\mathbb{R}^n)$. The Dunford–Pettis theorem [**11**, Theorem 4.30], i.e.

$$\{f_n\}_n \ \text{ is uniformly integrable in } \ L^1(\Omega) \ \Leftrightarrow \ \{f_n\}_n \\ \text{is relatively compact in the weak topology,}$$

implies that for every $k\in\mathbb{N}$ the sequence $\{\nabla T_k(u_s)\}_{s>0}$ is relatively compact in the weak topology of $L^1(\Omega;\mathbb{R}^n)$. As the set Ω is bounded, the Rellich–Kondrachov theorem [**11**, Theorem 9.16] for $W^{1,1}(\Omega)$ yields uniform integrability of the sequence $\{T_k(u_s)\}_{s>0}$ in the space $L^1(\Omega)$. Hence, there exists a function u such that

$$\begin{aligned} T_k(u_s) &\to T_k(u) \text{ strongly in } L^1(\Omega),\\ \nabla T_k(u_s) &\rightharpoonup \nabla T_k(u) \text{ weakly in } L^1(\Omega;\mathbb{R}^n). \end{aligned}$$

Thus, up to a subsequence, we have $u_s\to u$ in measure and almost everywhere, which gives (3.9). Additionally, the Dunford–Pettis theorem together with (3.2) imply that, up to a subsequence, we have

$$\nabla T_k(u_s) \xrightharpoonup{*} \nabla T_k(u) \text{ weakly-}* \text{ in } L_M(\Omega;\mathbb{R}^n). \tag{3.12}$$

Since $u_s\to u$ in measure, using (3.7) we obtain (3.10).

Now we focus on the last convergence in (3.11). For every $k \in \mathbb{N}$ we define

$$\mathcal{A}_{s,k} = \mathcal{A}(x, \nabla T_k(u_s)).$$

Using the other *a priori* estimate (3.3) and repeating the arguments from above we infer that, up to a subsequence, there exists $\mathcal{A}_k \in L_{M^*}(\Omega; \mathbb{R}^n)$ such that

$$\mathcal{A}_{s,k} \xrightarrow{*} \mathcal{A}_k \text{ weakly-} * \text{ in } L_{M^*}(\Omega; \mathbb{R}^n). \tag{3.13}$$

Step 5. Identification of the limit of $\mathcal{A}(x,\, \nabla T_k(u_s(x)))$.

We want to show that our limit obtained above in (3.13) is precisely of the form

$$\mathcal{A}_k = \mathcal{A}(x, \nabla T_k(u)) \text{ a.e. in } \Omega. \tag{3.14}$$

We are going to prove it via monotonicity trick. To use it, we must first show that

$$\int_\Omega (\mathcal{A}_k - \mathcal{A}(x,\eta)) \cdot (\nabla T_k(u) - \eta)\, \mathrm{d}x \geqslant 0 \quad \text{for every} \quad \eta \in \mathbb{R}^n. \tag{3.15}$$

We begin with showing that

$$\limsup_{s\to\infty} \int_\Omega \mathcal{A}_{s,k} \cdot \nabla T_k(u_s)\, \mathrm{d}x = \int_\Omega \mathcal{A}_k \cdot \nabla T_k(u)\, \mathrm{d}x. \tag{3.16}$$

Firstly, we consider a function $\Psi_l : \mathbb{R} \to [0,\, 1]$ defined as

$$\Psi_l(r) = \min\{(l + 1 - |r|)_+, 1\} \tag{3.17}$$

and using proposition 2.2, we can take an approximate sequence $\{\nabla(T_k(u))_\delta\}_\delta$ of smooth functions such that

$$\nabla(T_k(u))_\delta \xrightarrow[\delta\to 0]{M} \nabla T_k(u) \text{ modularly in } L_M(\Omega; \mathbb{R}^n).$$

Having this in mind, we will show that

$$\lim_{\delta\to 0} \limsup_{s\to\infty} \int_\Omega \mathcal{A}_{s,k} \cdot \nabla\big(T_k(u_s) - (T_k(u))_\delta\big)\, \mathrm{d}x = 0. \tag{3.18}$$

The condition $(A2)$ for the operator $\mathcal{A}$ implies $\mathcal{A}(x,\, 0) = 0$. Hence, for $l \geqslant k$ this observation yields

$$\begin{aligned}
&\int_\Omega \mathcal{A}_{s,k} \cdot \nabla\big(T_k(u_s) - (T_k(u))_\delta\big) \Psi_l(u_s)\, \mathrm{d}x \\
&\quad = \int_\Omega \mathcal{A}_{s,k} \cdot \nabla\big(T_k(u_s) - (T_k(u))_\delta\big)\, \mathrm{d}x \\
&\qquad + \int_{\{|u_s|>l\}} \mathcal{A}(x,0) \cdot \nabla\big(0 - (T_k(u))_\delta\big)(\Psi_l(u_s) - 1)\, \mathrm{d}x \\
&\quad = \int_\Omega \mathcal{A}_{s,k} \cdot \nabla\big(T_k(u_s) - (T_k(u))_\delta\big)\, \mathrm{d}x.
\end{aligned}$$

https://doi.org/10.1017/prm.2023.113 Published online by Cambridge University Press

Therefore, (3.18) is equivalent to

$$\lim_{l\to\infty}\lim_{\delta\to 0}\limsup_{s\to\infty}\int_\Omega \mathcal{A}_{s,k}\cdot\nabla\big(T_k(u_s)-(T_k(u))_\delta\big)\Psi_l(u_s)\,\mathrm{d}x=0. \tag{3.19}$$

Now we notice that it is enough to have

$$\lim_{l\to\infty}\lim_{\delta\to 0}\limsup_{s\to\infty}\int_\Omega \mathcal{A}_{s,l+1}\cdot\nabla\big(T_k(u_s)-(T_k(u))_\delta\big)\Psi_l(u_s)\,\mathrm{d}x=0. \tag{3.20}$$

Indeed, having this result, (3.19) will be satisfied if we manage to prove that for $l\geqslant k$,

$$\begin{aligned}J:&=\int_\Omega\big(\mathcal{A}_{s,k}-\mathcal{A}_{s,l+1}\big)\cdot\nabla\big(T_k(u_s)-(T_k(u))_\delta\big)\Psi_l(u_s)\,\mathrm{d}x\\&=\int_\Omega\big(\mathcal{A}_{s,l+1}-\mathcal{A}(x,0)\big)\cdot\nabla(T_k(u))_\delta\mathbb{1}_{\{k<|u_s|\}}\Psi_l(u_s)\,\mathrm{d}x\\&=\int_\Omega\mathcal{A}_{s,l+1}\cdot\nabla(T_k(u))_\delta\mathbb{1}_{\{k<|u_s|\}}\Psi_l(u_s)\,\mathrm{d}x\end{aligned}$$

tends to zero as $s\to\infty$ and $\delta\to 0$. In order to do so, we just need to prove that

$$\begin{aligned}\lim_{\delta\to 0}\limsup_{s\to\infty}|J|&\leqslant\lim_{\delta\to 0}\limsup_{s\to\infty}\int_\Omega|\mathcal{A}_{s,l+1}|\mathbb{1}_{\{k<|u_s|\}}\Psi_l(u_s)|\nabla(T_k(u))_\delta|\,\mathrm{d}x\\&\leqslant\lim_{\delta\to 0}\int_\Omega|\mathcal{A}_{l+1}|\mathbb{1}_{\{k<|u|\}}\Psi_l(u)|\nabla(T_k(u))_\delta|\,\mathrm{d}x\\&=\int_\Omega|\mathcal{A}_{l+1}|\mathbb{1}_{\{k<|u|\}}\Psi_l(u)|\nabla T_k(u)|\,\mathrm{d}x=0.\end{aligned} \tag{3.21}$$

For the limit as $s\to\infty$ we will use lemma 2.8 with

$$z_s:=|\mathcal{A}_{s,l+1}|\cdot|\nabla(T_k(u))_\delta|\overset{s\to\infty}{\rightharpoonup}|\mathcal{A}_{l+1}|\cdot|\nabla(T_k(u))_\delta|=z\qquad\text{weakly in }L^1(\Omega)$$

and $w_s=\Psi_l(u_s)\mathbb{1}_{\{k<|u_s|\}}$. The convergence $z_s\rightharpoonup z$ in $L^1(\Omega)$ is a consequence of (3.13). And (3.9) gives us that $w_s\to w=\Psi_l(u)\mathbb{1}_{\{k<|u_s|\}}$ a.e. in Ω. The external limit with $\delta\to 0$ arises from modular convergence in (3.21). In addition, since we have

$$\nabla T_k(u)\mathbb{1}_{\{k<|u\}}=0,$$

the last equality in (3.21) follows.

Now, to obtain (3.20), we test (3.1) by the sequence

$$\varphi := \Psi_l(u_s)\big(T_k(u_s) - (T_k(u))_\delta\big),$$

where Ψ_l is defined in (3.17). Thus, we have

$$\begin{aligned}
&\int_\Omega \mathcal{A}(x,\nabla u_s)\cdot\nabla\big(\Psi_l(u_s)(T_k(u_s)-(T_k(u))_\delta)\big)\,\mathrm{d}x\\
&\quad+\int_\Omega \Phi_s(u_s)\cdot\nabla\big(\Psi_l(u_s)(T_k(u_s)-(T_k(u))_\delta)\big)\,\mathrm{d}x\\
&\quad+\int_\Omega b(x,u_s)\Psi_l(u_s)\big(T_k(u_s)-(T_k(u))_\delta\big)\,\mathrm{d}x\\
&=\int_\Omega T_s(f)\Psi_l(u_s)\big(T_k(u_s)-(T_k(u))_\delta\big)\,\mathrm{d}x\\
&\quad+\int_\Omega F\cdot\nabla\big(\Psi_l(u_s)(T_k(u_s)-(T_k(u))_\delta)\big)\,\mathrm{d}x.
\end{aligned}\tag{3.22}$$

Firstly, we consider the first term in the right-hand side of (3.22). Since we know that

$$u_s \to u \text{ a.e. in } \Omega,$$

we would like to use the Lebesgue-dominated convergence theorem. To do so, we note that

$$\begin{aligned}
&\lim_{\delta\to 0}\lim_{s\to\infty}\left|\int_\Omega T_s(f)\Psi_l(u_s)\big(T_k(u_s)-(T_k(u))_\delta\big)\,\mathrm{d}x\right|\\
&\leqslant \lim_{\delta\to 0}\lim_{s\to\infty}\int_\Omega \big|T_s(f)\big|\Psi_l(u_s)\big|\big(T_k(u_s)-T_k(u)\big)\big|\,\mathrm{d}x\\
&\quad+\lim_{\delta\to 0}\lim_{s\to\infty}\int_\Omega \big|T_s(f)\big|\Psi_l(u_s)\big|\big(T_k(u)-(T_k(u))_\delta\big)\big|\,\mathrm{d}x\\
&\leqslant \lim_{\delta\to 0}\lim_{s\to\infty}\int_\Omega 2k|f|\,\mathrm{d}x+\lim_{\delta\to 0}\lim_{s\to\infty}\int_\Omega |f|\cdot|T_k(u)-(T_k(u))_\delta|\,\mathrm{d}x\\
&=2k||f||_{L^1(\Omega)}+\lim_{\delta\to 0}\int_\Omega |f|\cdot|T_k(u)-(T_k(u))_\delta|\,\mathrm{d}x.
\end{aligned}$$

Using the modular approximation result (see proposition 2.2) we get $|(T_k(u))_\delta|\leqslant ck$, which implies

$$|T_k(u)-(T_k(u))_\delta|\leqslant(1+c)k. \tag{3.23}$$

Thus, we infer that

$$\lim_{l\to\infty}\lim_{\delta\to 0}\limsup_{s\to\infty}\int_\Omega T_s(f)\Psi_l(u_s)\big(T_k(u_s)-(T_k(u))_\delta\big)\,\mathrm{d}x=0. \tag{3.24}$$

https://doi.org/10.1017/prm.2023.113 Published online by Cambridge University Press

Secondly, for the second term in the right-hand side of (3.22), we define

$$\begin{aligned}
&\int_\Omega F\cdot\nabla\big(\Psi_l(u_s)(T_k(u_s)-(T_k(u))_\delta)\big)\,\mathrm{d}x\\
&=\int_\Omega F\cdot\nabla\Psi_l(u_s)\big(T_k(u_s)-(T_k(u))_\delta\big)\,\mathrm{d}x+\int_\Omega F\cdot\Psi_l(u_s)\nabla T_k(u_s)\,\mathrm{d}x\\
&\quad-\int_\Omega F\cdot\Psi_l(u_s)\nabla(T_k(u))_\delta\,\mathrm{d}x\\
&=: h_1+h_2-h_3.
\end{aligned}$$

Thanks to the Fenchel–Young inequality and (3.23), we get

$$\begin{aligned}
&\lim_{l\to\infty}\lim_{\delta\to0}\limsup_{s\to\infty}|h_1|\\
&=\lim_{l\to\infty}\lim_{\delta\to0}\limsup_{s\to\infty}\Big|\int_\Omega F\cdot\nabla\Psi_l(u_s)\big(T_k(u_s)-(T_k(u))_\delta\big)\,\mathrm{d}x\Big|\\
&\leqslant C\lim_{l\to\infty}\lim_{\delta\to0}\limsup_{s\to\infty}\int_{\{l\leqslant|u_s|\leqslant l+1\}}\big|F\cdot\nabla T_{l+1}(u_s)\big|\,\mathrm{d}x\\
&\leqslant C\lim_{l\to\infty}\limsup_{s\to\infty}\left(\int_{\{l<|u_s|<l+1\}}\frac{1}{4}M\big(x,c_1^{\mathcal{A}}\nabla T_{l+1}(u_s)\big)\,\mathrm{d}x\right.\\
&\quad\left.+\int_{\{l<|u_s|<l+1\}}M^*\left(x,\frac{4}{c_1^{\mathcal{A}}}F\right)\,\mathrm{d}x\right)\\
&\leqslant C\lim_{l\to\infty}\gamma\left(\frac{l}{m_1(c_1l)}\right)=0.
\end{aligned}\tag{3.25}$$

where in the last line we use (3.4), (3.8) and the fact that m_1 is superlinear at infinity as an N-function. Note that

$$\nabla T_k(u_s)\overset{s\to\infty}{\rightharpoonup}\nabla T_k(u)\ \text{weakly in}\ L^1(\Omega;\mathbb{R}^n),$$

combining with the fact

$$\begin{aligned}
\int_\Omega|F\nabla T_k(u_s)|\,\mathrm{d}x&\leqslant\int_\Omega M^*\left(x,\frac{1}{c_1^{\mathcal{A}}}F\right)\,\mathrm{d}x+\int_\Omega M(x,c_1^{\mathcal{A}}\nabla T_{l+1}(u_s))\,\mathrm{d}x\\
&\leqslant C,
\end{aligned}$$

where C is independent of s, we deduce that

$$F\nabla T_k(u_s)\overset{s\to\infty}{\rightharpoonup}F\nabla T_k(u)\quad\text{weakly in}\ L^1(\Omega).$$

Recall that $|\Psi_l(u_s)|\leqslant 1$ and $\Psi_l(u_s)\xrightarrow[s\to\infty]{a.e.}\Psi_l(u)$, it follows from lemma 2.8 that

$$\lim_{s\to\infty}\int_\Omega F\Psi_l(u_s)\nabla T_k(u_s)\,\mathrm{d}x=\int_\Omega F\Psi_l(u)\nabla T_k(u)\,\mathrm{d}x.$$

Notice that by definition of the function Ψ_l (see (3.17)), we have

$$\Psi_l(u) \xrightarrow{l\to\infty} 1 \text{ a.e. in } \Omega. \tag{3.26}$$

Thus, collecting all the facts mentioned above and using the Lebesgue-dominated convergence theorem we infer that

$$\begin{aligned}\lim_{l\to\infty}\lim_{s\to\infty} h_2 &= \lim_{l\to\infty}\lim_{s\to\infty}\int_\Omega F\cdot\Psi_l(u_s)\nabla T_k(u_s)\,\mathrm{d}x\\ &= \lim_{l\to\infty}\int_\Omega F\cdot\Psi_l(u)\nabla T_k(u)\,\mathrm{d}x\\ &= \int_\Omega F\cdot\nabla T_k(u)\,\mathrm{d}x.\end{aligned}$$

Additionally, by the Lebesgue-dominated convergence theorem and the fact $\nabla(T_k(u))_\delta \to \nabla T_k(u)$ modularly in $L_M(\Omega;\mathbb{R}^n)$, we see that

$$\begin{aligned}\lim_{l\to\infty}\lim_{\delta\to 0}\lim_{s\to\infty} h_3 &= \lim_{l\to\infty}\lim_{\delta\to 0}\lim_{s\to\infty}\int_\Omega F\cdot\Psi_l(u_s)\nabla(T_k(u))_\delta\,\mathrm{d}x\\ &= \lim_{l\to\infty}\lim_{\delta\to 0}\int_\Omega F\cdot\Psi_l(u)\nabla(T_k(u))_\delta\,\mathrm{d}x.\\ &= \int_\Omega F\cdot\nabla T_k(u)\,\mathrm{d}x.\end{aligned}$$

All in all, we conclude that

$$\begin{aligned}&\lim_{l\to\infty}\lim_{\delta\to 0}\limsup_{s\to\infty}\int_\Omega F\cdot\nabla\big(\Psi_l(u_s)(T_k(u_s)-(T_k(u))_\delta)\big)\,\mathrm{d}x\\ &\quad= \lim_{l\to\infty}\lim_{\delta\to 0}\limsup_{s\to\infty}\big(h_1+h_2-h_3\big) = 0.\end{aligned} \tag{3.27}$$

Now, we will focus on the left-hand side of (3.22). Let us denote

$$\begin{aligned}&\int_\Omega \mathcal{A}(x,\nabla u_s)\cdot\nabla\big(\Psi_l(u_s)(T_k(u_s)-(T_k(u))_\delta)\big)\,\mathrm{d}x\\ &\quad+\int_\Omega \Phi_s(u_s)\cdot\nabla\big(\Psi_l(u_s)(T_k(u_s)-(T_k(u))_\delta)\big)\,\mathrm{d}x\\ &\quad+\int_\Omega b(x,u_s)\Psi_l(u_s)\big(T_k(u_s)-(T_k(u))_\delta\big)\,\mathrm{d}x\\ &=: I_1+I_2+I_3.\end{aligned} \tag{3.28}$$

At first we concentrate on the easier terms. We are going to show that both

$$\lim_{\delta\to 0}\lim_{s\to\infty}(I_2) = 0 \text{ and } \lim_{\delta\to 0}\lim_{s\to\infty}(I_3) = 0.$$

Indeed, for I_3 we have

$$I_3 = \int_\Omega b(x,T_{l+1}(u_s))\Psi_l(u_s)\big(T_k(u_s)-(T_k(u))_\delta\big)\,\mathrm{d}x,$$

https://doi.org/10.1017/prm.2023.113 Published online by Cambridge University Press

due to the definition of the function Ψ_l (see (3.17)). Thanks to the assumption (b), we know that $b(\cdot,\, s) \in L^1(\Omega)$ for each $s \in \mathbb{R}$. This fact, together with (3.23) yields

$$\begin{aligned}
\lim_{\delta\to 0}\lim_{s\to\infty} |I_3| &= \lim_{\delta\to 0}\lim_{s\to\infty} \left| \int_\Omega b(x, T_{l+1}(u_s))\Psi_l(u_s)\big(T_k(u_s) - (T_k(u))_\delta\big)\,\mathrm{d}x \right| \\
&\leqslant \lim_{\delta\to 0}\lim_{s\to\infty} \int_\Omega |b(x, T_{l+1}(u_s))|\Psi_l(u_s)\big|T_k(u_s) - T_k(u)\big|\,\mathrm{d}x \\
&\quad + \lim_{\delta\to 0}\lim_{s\to\infty} \int_\Omega |b(x, T_{l+1}(u_s))|\Psi_l(u_s)\big|T_k(u) - (T_k(u))_\delta\big|\,\mathrm{d}x \\
&=: I_3^1 + I_3^2.
\end{aligned} \tag{3.29}$$

Now, from the Lebesgue-dominated convergence theorem we have

$$\lim_{\delta\to 0}\lim_{s\to\infty}(I_3^1) = \lim_{\delta\to 0}\lim_{s\to\infty}(I_3^2) = 0.$$

To justify the convergence of I_2, note that by definition of Ψ_l and the chain rule we can also rewrite it as

$$\begin{aligned}
I_2 &= \int_\Omega \Phi(T_s(u_s)) \cdot \nabla\big(T_k(u_s) - (T_k(u))_\delta\big)\Psi_l(u_s)\,\mathrm{d}x \\
&\quad + \int_\Omega \Phi(T_s(u_s)) \cdot \nabla\Psi_l(u_s)(T_k(u_s) - (T_k(u))_\delta)\,\mathrm{d}x \\
&=: I_2^1 + I_2^2.
\end{aligned}$$

For $s \geqslant l+1$, we have

$$I_2^1 = \int_\Omega \Phi(T_{l+1}(u_s)) \cdot \nabla\big(T_k(u_s) - (T_k(u))_\delta\big)\Psi_l(u_s)\,\mathrm{d}x.$$

Since Φ is continuous and $u_s \to u$ almost everywhere in Ω, we obtain

$$\Psi_l(u_s)\Phi(T_{l+1}(u_s)) \to \Psi_l(u)\Phi(T_{l+1}(u)) \quad \text{a.e. in} \quad \Omega.$$

As $\Phi(T_{l+1}(u_s))$ is uniformly bounded with respect to s, i.e.

$$\|\Phi(T_{l+1}(u_s))\|_{L^\infty(\Omega;\mathbb{R}^n)} \leqslant \sup_{\tau\in[-l-1,l+1]} |\Phi(\tau)| < C,$$

where the constant $C > 0$ is independent of $s \in \mathbb{N}$ and as the following facts

$$\begin{aligned}
|\Psi_l(u_s)| &\leqslant 1 && \text{a.e. in } \Omega, \\
\nabla T_k(u_s) &\rightharpoonup \nabla T_k(u) \text{weakly in } L^1(\Omega;\mathbb{R}^n), \\
\nabla(T_k(u))_\delta &\xrightarrow[\delta\to 0]{M} \nabla T_k(u) \text{modularly in } L_M(\Omega;\mathbb{R}^n),
\end{aligned}$$

it follows from lemma 2.8 that

$$\lim_{\delta\to 0}\limsup_{s\to\infty} I_2^1 = 0.$$

https://doi.org/10.1017/prm.2023.113 Published online by Cambridge University Press

Let us write

$$I_2^2 = \int_\Omega \operatorname{div}\left(\int_0^{T_{l+1}(u_s)} \Phi(r)\Psi_l'(r)\,\mathrm{d}r\right)\big(T_k(u_s) - (T_k(u))_\delta\big)\,\mathrm{d}x,$$

we may use the Gauss–Green theorem and obtain

$$I_2^2 = -\int_\Omega \int_0^{T_{l+1}(u_s)} \Phi(r)\Psi_l'(r)\,\mathrm{d}r \cdot \nabla\big(T_k(u_s) - (T_k(u))_\delta\big)\,\mathrm{d}x.$$

Using the same arguments as above, we infer that

$$\lim_{\delta\to 0}\limsup_{s\to\infty} I_2^2 = 0.$$

Therefore, we have

$$\lim_{l\to\infty}\lim_{\delta\to 0}\limsup_{s\to\infty} I_2 = 0.$$

Finally, we will concentrate on the most challenging and difficult term I_1. As before, we rewrite it as follows:

$$\begin{aligned}
I_1 &= \int_\Omega \mathcal{A}(x,\nabla u_s)\cdot\nabla\left(\Psi_l(u_s)(T_k(u_s) - (T_k(u))_\delta)\right)\mathrm{d}x \\
&= \int_\Omega \mathcal{A}(x,\nabla u_s)\cdot\nabla\Psi_l(u_s)\big(T_k(u_s) - (T_k(u))_\delta\big)\,\mathrm{d}x \\
&\quad + \int_\Omega \mathcal{A}(x,\nabla u_s)\cdot\nabla\big(T_k(u_s) - (T_k(u))_\delta\big)\Psi_l(u_s)\,\mathrm{d}x \\
&=: I_1^1 + I_1^2.
\end{aligned} \tag{3.30}$$

To estimate I_1^1, we will use (3.23) and (3.4) to get

$$\begin{aligned}
&\lim_{l\to\infty}\lim_{\delta\to 0}\limsup_{s\to\infty} |I_1^1| \\
&\quad \leqslant \lim_{l\to\infty}\left(\lim_{\delta\to 0}\limsup_{s\to\infty}\int_{\{l<|u_s|<l+1\}} \big|\mathcal{A}(x,\nabla u_s)\cdot\nabla u_s\big|\big|\big(T_k(u_s) - (T_k(u))_\delta\big)\big|\,\mathrm{d}x\right) \\
&\quad \leqslant C\lim_{l\to\infty}\left(\lim_{\delta\to 0}\limsup_{s\to\infty}\int_{\{l<|u_s|<l+1\}} \big|\mathcal{A}(x,\nabla u_s)\cdot\nabla u_s\big|\,\mathrm{d}x\right) \\
&\quad = C\lim_{l\to\infty}\left(\limsup_{s\to\infty}\int_{\{l<|u_s|<l+1\}} \mathcal{A}(x,\nabla u_s)\cdot\nabla u_s\,\mathrm{d}x\right) \\
&\quad \leqslant C\lim_{l\to\infty}\gamma\left(\frac{l}{m_1(c_1 l)}\right) = 0.
\end{aligned}$$

https://doi.org/10.1017/prm.2023.113 Published online by Cambridge University Press

Now we notice that (3.30) yields

$$\begin{aligned}
&\lim_{l\to\infty}\lim_{\delta\to 0}\limsup_{s\to\infty}\left(I_1^2\right)\\
&\qquad=\lim_{l\to\infty}\lim_{\delta\to 0}\limsup_{s\to\infty}\int_\Omega \mathcal{A}(x,\nabla u_s)\cdot\nabla\big(T_k(u_s)-(T_k(u))_\delta\big)\Psi_l(u_s)\,\mathrm{d}x=0.
\end{aligned}\tag{3.31}$$

Hence, by virtue of all the above limits, (3.31) is actually equivalent to (3.20). Thus, we arrive at (3.18). According to (3.18) and (3.13), we obtain

$$\begin{aligned}
\lim_{\delta\to 0}\limsup_{s\to\infty}\int_\Omega \mathcal{A}_{s,k}\cdot\nabla T_k(u_s)\,\mathrm{d}x &= \lim_{\delta\to 0}\limsup_{s\to\infty}\int_\Omega \mathcal{A}_{s,k}\cdot\nabla\big(T_k(u)\big)_\delta\,\mathrm{d}x\\
&=\lim_{\delta\to 0}\int_\Omega \mathcal{A}_k\cdot\nabla\big(T_k(u)\big)_\delta\,\mathrm{d}x\\
&=\int_\Omega \mathcal{A}_k\cdot\nabla T_k(u)\,\mathrm{d}x.
\end{aligned}\tag{3.32}$$

Eventually, we get (3.16).

We are about to complete the proof of identification of the limit of $\{\mathcal{A}_{s,k}\}_{s>0}$. Using the monotonicity of $\mathcal{A}$ (condition (A3)) we infer that for every $\eta\in L^\infty(\Omega;\mathbb{R}^n)$ we have

$$\int_\Omega \mathcal{A}_{s,k}\cdot\eta\,\mathrm{d}x+\int_\Omega \mathcal{A}(x,\eta)\cdot(\nabla T_k(u_s)-\eta)\,\mathrm{d}x\leqslant\int_\Omega \mathcal{A}_{s,k}\cdot\nabla T_k(u_s)\,\mathrm{d}x. \tag{3.33}$$

Because of (3.12), (3.13) and (3.16), we may take the upper limit with $s\to\infty$ of both sides of (3.33) to get

$$\int_\Omega \mathcal{A}_k\cdot\eta\,\mathrm{d}x+\int_\Omega \mathcal{A}(x,\eta)\cdot(\nabla T_k(u)-\eta)\,\mathrm{d}x\leqslant\int_\Omega \mathcal{A}_k\cdot\nabla T_k(u)\,\mathrm{d}x,$$

which, by rearranging terms is obviously equivalent to (3.15). Hence, we may apply the famous monotonicity trick (lemma 2.7), which ends the proof of this step.

Step 6. Renormalized solutions

Now our goal is to show the existence of renormalized solutions, which will end the proof of theorem 1.3. In fact, we are going to show that the function u, obtained as a limit in step 4 is precisely the renormalized solution. By definition, we must check whether the three conditions $(R1)$, $(R2)$ and $(R3)$ are satisfied.

Condition $(R1)$.

We just notice that thanks to the convergence in (3.11), condition $(R1)$ is satisfied.

Condition $(R2)$.

As we know that $T_k(u)\in V_0^1L_M(\Omega)$, proposition 2.2 yields existence of a sequence $\{u_r\}_{r>0}\subset C_c^\infty(\Omega)$ for which we have

$$\begin{aligned}
&u_r\to u \text{ a.e. in } \Omega,\\
&\nabla T_k(u_r)\xrightarrow{*}\nabla T_k(u)\text{weakly-}*\text{ in } L_M(\Omega;\mathbb{R}^n),\\
&\nabla h(u_r)\rightharpoonup\nabla h(u)\text{weakly in } L_M(\Omega;\mathbb{R}^n),
\end{aligned}\tag{3.34}$$

https://doi.org/10.1017/prm.2023.113 Published online by Cambridge University Press

for an arbitrary function $h \in C_c^1(\Omega)$. Now, for such a fixed h and $\phi \in W_0^{1,\infty}(\Omega)$ we test (3.1) by $\Psi_l(u_s)h(u_r)\phi$, where Ψ_l defined in (3.17). Therefore, we get

$$\begin{aligned}
&\int_\Omega \mathcal{A}(x,\nabla u_s)\cdot\nabla\big(\Psi_l(u_s)h(u_r)\phi\big)\,\mathrm{d}x + \int_\Omega \Phi_s(u_s)\cdot\nabla\big(\Psi_l(u_s)h(u_r)\phi\big)\,\mathrm{d}x \\
&\quad + \int_\Omega b(x,u_s)\Psi_l(u_s)h(u_r)\phi\,\mathrm{d}x \qquad\qquad (3.35)\\
&= \int_\Omega T_s(f)\Psi_l(u_s)h(u_r)\phi\,\mathrm{d}x + \int_\Omega F\cdot\nabla\big(\Psi_l(u_s)h(u_r)\phi\big)\,\mathrm{d}x.
\end{aligned}$$

Let us denote the terms on the left-hand side of the equation above as $L^1_{s,r,l}$, $L^2_{s,r,l}$ and $L^3_{s,r,l}$ respectively. Also, write

$$L^1_{s,r,l} + L^2_{s,r,l} + L^3_{s,r,l} = R^1_{s,r,l} + R^2_{s,r,l},$$

where $R^1_{s,r,l}$, $R^2_{s,r,l}$ stand for the right-hand side of (3.35) respectively.

At first, we see that by the Lebesgue-dominated convergence theorem, we get

$$\lim_{l\to\infty}\lim_{r\to\infty}\lim_{s\to\infty} R^1_{s,r,l} = \int_\Omega f h(u)\phi\,\mathrm{d}x.$$

For the second term in the right-hand side of (3.35), we can write

$$\begin{aligned}
R^2_{s,r,l} &= \int_\Omega F\cdot\nabla\big(\Psi_l(u_s)h(u_r)\phi\big)\,\mathrm{d}x \\
&= \int_\Omega F\cdot\nabla\big(\Psi_l(u_s)\big)h(u_r)\phi\,\mathrm{d}x + \int_\Omega F\cdot\nabla\big(h(u_r)\phi\big)\Psi_l(u_s)\,\mathrm{d}x.
\end{aligned}$$

By the similar arguments to (3.25), we have

$$\begin{aligned}
&\lim_{l\to\infty}\lim_{r\to\infty}\limsup_{s\to\infty}\Big|\int_\Omega F\cdot\nabla\big(\Psi_l(u_s)\big)h(u_r)\phi\,\mathrm{d}x\Big| \\
&\quad\leqslant \|h\|_{L^\infty(\Omega)}\|\phi\|_{L^\infty(\Omega)}\lim_{l\to\infty}\limsup_{s\to\infty}\int_{\{l\leqslant u_s\leqslant l+1\}}|F|\cdot|\nabla T_{l+1}(u_s)|\,\mathrm{d}x \\
&\quad\leqslant C\lim_{l\to\infty}\gamma\left(\frac{l}{m_1(c_1 l)}\right) = 0.
\end{aligned}$$

Moreover, since $F \in E_{M^*}(\Omega;\mathbb{R}^n) \subseteq L^1(\Omega;\mathbb{R}^n)$, $\Psi_l(u_s)$ converges to $\Psi_l(u)$ a.e. in Ω with $|\Psi_l(u_s)| \leqslant 1$, the product sequence $F\Psi_l(u_s)$ also converges strongly to $F\Psi_l(u)$ in $L^1(\Omega;\mathbb{R}^n)$ as $s\to\infty$. Combining with the following facts

$$\nabla(h(u_r)\phi) \in L^\infty(\Omega;\mathbb{R}^n), \quad \nabla\big(h(u_r)\phi\big) \overset{*}{\rightharpoonup} \nabla\big(h(u)\phi\big) \quad \text{weakly-* in } L_M(\Omega;\mathbb{R}^n),$$

we deduced that

$$\begin{aligned}
\lim_{r\to\infty}\lim_{s\to\infty}\int_\Omega F\Psi_l(u_s)\cdot\nabla\big(h(u_r)\phi\big)\,\mathrm{d}x &= \lim_{r\to\infty}\int_\Omega F\Psi_l(u)\cdot\nabla\big(h(u_r)\phi\big)\,\mathrm{d}x \\
&= \int_\Omega F\Psi_l(u)\cdot\nabla\big(h(u)\phi\big)\,\mathrm{d}x.
\end{aligned}$$

https://doi.org/10.1017/prm.2023.113 Published online by Cambridge University Press

Notice that there exists $m > 0$ such that $\operatorname{supp}(h) \subset [-m, m]$. Choosing such m, we may interchange T_{l+1} with T_m. Then $\Psi_l(u) = \Psi_l(T_m(u)) = 1$ for $l > m$. Thus, we have

$$\begin{aligned}\lim_{l\to\infty}\lim_{r\to\infty}\lim_{s\to\infty} R^2_{s,r,l} &= \lim_{l\to\infty}\int_\Omega F\Psi_l(u)\nabla\big(h(u)\phi\big)\,\mathrm{d}x\\ &= \int_\Omega F\nabla\big(h(u)\phi\big)\,\mathrm{d}x.\end{aligned}$$

Next, we concentrate on the left-hand side of (3.35).

Firstly, we look at the term $L^3_{s,r,l}$. Thanks to assumption (b), we obtained that $b(x, T_{l+1}(u_s)) \to b(x, T_{l+1}(u))$ almost everywhere in Ω, and $\{b(\cdot, T_{l+1}(u_s)\}$ is uniformly integrable. As Ω has a finite measure, by Vitali convergence theorem, we have

$$b\big(\cdot, T_{l+1}(u_s)\big) \to b\big(\cdot, T_{l+1}(u)\big) \text{ in } L^1(\Omega).$$

This combined with the facts that $\Psi_l(u_s)$ converges to $\Psi_l(u)$ a.e. in Ω and $|\Psi_l(u_s)| \leqslant 1$, imply that the product sequence $b(x, u_s)\Psi_l(u_s) = b(x, T_{l+1}(u_s))\Psi_l(u_s)$ also converges strongly to $b(x, u)\Psi_l(u) = b(x, T_{l+1}(u))\Psi_l(u)$ in $L^1(\Omega)$. Since the term $h(u_r)\phi$ is bounded, we obtain

$$\lim_{s\to\infty}\int_\Omega b(x,u_s)\Psi_l(u_s)h(u_r)\phi\,\mathrm{d}x = \int_\Omega b(x,u)\Psi_l(u)h(u_r)\phi\,\mathrm{d}x.$$

Moreover, $h(u_r) \to h(u)$ a.e. in Ω, which leads us to

$$\lim_{r\to\infty}\int_\Omega b(x,u)\Psi_l(u)h(u_r)\phi\,\mathrm{d}x = \int_\Omega b(x,u)\Psi_l(u)h(u)\phi\,\mathrm{d}x.$$

For $l \geqslant m$, where m is such that $\operatorname{supp}(h) \subset [-m, m]$, we infer that

$$\lim_{l\to\infty}\lim_{r\to\infty}\lim_{s\to\infty} L^3_{s,r,l} = \int_\Omega b(x,u)h(u)\phi\,\mathrm{d}x.$$

Secondly, for the term $L^2_{s,r,l}$, choosing $s \geqslant l+1$, we can rewrite it as follows

$$\begin{aligned}L^2_{s,r,l} &= \int_\Omega \Phi(T_{l+1}(u_s))\cdot\nabla\big(h(u_r)\phi\big)\Psi_l(u_s)\,\mathrm{d}x\\ &\quad+\int_\Omega \Phi(T_{l+1}(u_s))\cdot\Psi_l'(u_s)\nabla T_{l+1}(u_s)(h(u_r)\phi)\,\mathrm{d}x\\ &=: L^{2,1}_{s,r,l} + L^{2,2}_{s,r,l}.\end{aligned}\tag{3.36}$$

As $\Phi(T_{l+1}(u_s))\Psi_l(u_s)$ is uniformly bounded, the *a.e.* convergence of $\{u_s\}_{s>0}$ and the Vitali theorem provide that $\Phi(T_{l+1}(u_s))\Psi_l(u_s) \to \Phi(T_{l+1}(u))\Psi_l(u)$ in $L^1(\Omega;\mathbb{R}^n)$, thus

$$\lim_{s\to\infty}\int_\Omega \Phi(T_{l+1}(u_s))\cdot\nabla\big(h(u_r)\phi\big)\Psi_l(u_s)\,\mathrm{d}x = \int_\Omega \Phi(T_{l+1}(u))\cdot\nabla\big(h(u_r)\phi\big)\Psi_l(u)\,\mathrm{d}x.$$

Since $\nabla(h(u_r)\phi) \xrightarrow{*} \nabla(h(u)\phi)$ in $L_M(\Omega;\mathbb{R}^n)$ and Φ is Lipschitz continuous, we find that

$$\lim_{r\to\infty}\int_\Omega \Phi(T_{l+1}(u))\cdot\nabla\big(h(u_r)\phi\big)\Psi_l(u)\,\mathrm{d}x = \int_\Omega \Phi(T_{l+1}(u))\cdot\nabla\big(h(u)\phi\big)\Psi_l(u)\,\mathrm{d}x. \tag{3.37}$$

For $l \geqslant m$, where m is such that $\mathrm{supp}(h) \subset [-m, m]$. Rewriting (3.37), we arrive at

$$\lim_{l\to\infty}\lim_{r\to\infty}\lim_{s\to\infty} L^{2,1}_{s,r,l} = \int_\Omega \Phi(u)\cdot\nabla(h(u)\phi)\,\mathrm{d}x.$$

Now we concentrate on the second term from (3.36). Again, similarly as before we may rewrite it as follows

$$L^{2,2}_{s,r,l} = \int_\Omega \mathrm{div}\left(\int_0^{T_{l+1}(u_s)} \Phi(t)\Psi_l'(t)\,\mathrm{d}t\right) h(u_r)\phi\,\mathrm{d}x.$$

and using the Gauss–Green theorem, we obtain

$$L^{2,2}_{s,r,l} = -\int_\Omega \int_0^{T_{l+1}(u_s)} \Phi(t)\Psi_l'(t)\,\mathrm{d}t\cdot\nabla(h(u_r)\phi)\,\mathrm{d}x.$$

For the limit with $s\to\infty$, we observe that

$$\begin{aligned}
\left|L^{2,2}_{s,r,l}\right| &= \left|\int_\Omega \int_0^{T_{l+1}(u_s)} \Phi(t)\Psi_l'(t)\,\mathrm{d}t\cdot\nabla(h(u_r)\phi)\,\mathrm{d}x\right| \\
&\leqslant \int_\Omega \left|\int_0^{T_{l+1}(u_s)} \Phi(t)\Psi_l'(t)\,\mathrm{d}t\cdot\nabla(h(u_r)\phi)\right|\mathrm{d}x \\
&\leqslant \int_\Omega \left|\int_0^{T_{l+1}(u_s)} \Phi(t)\Psi_l'(t)\,\mathrm{d}t\right|\cdot\left|\nabla(h(u_r)\phi)\right|\mathrm{d}x \\
&\leqslant \int_\Omega \sqrt{n}\left(\sup_{i\in\{1,\dots,n\}}\left|\int_0^{T_{l+1}(u_s)} \Phi_i(t)\Psi_l'(t)\,\mathrm{d}t\right|\right)\cdot\left|\nabla(h(u_r)\phi)\right|\mathrm{d}x \\
&\leqslant \int_\Omega 2\sqrt{n}(l+1)\left(\sup_{i\in\{1,\dots,n\}}\sup_{y\in[-l-1,l+1]}|\Phi_i(y)|\right)\cdot\left|\nabla(h(u_r)\phi)\right|\mathrm{d}x.
\end{aligned}$$

Since the term $\nabla(h(u_r)\phi)$ is bounded, $\Phi = (\Phi_1, \Phi_2, \dots \Phi_n)$ is Lipschitz and $u_s \to u$ almost everywhere in Ω, by the Lebesgue-dominated convergence theorem, we infer

https://doi.org/10.1017/prm.2023.113 Published online by Cambridge University Press

that

$$\lim_{s\to\infty} L^{2,2}_{s,r,l} = -\int_\Omega \int_0^{T_{l+1}(u)} \Phi(t)\Psi_l'(t)\,\mathrm{d}t \cdot \nabla(h(u_r)\phi)\,\mathrm{d}x$$
$$= \int_\Omega \Phi(T_{l+1}(u)) \cdot \nabla T_{l+1}(u)\Psi_l'(u)(h(u_r)\phi)\,\mathrm{d}x.$$

Using the Lebesgue-dominated convergence theorem again, we obtain

$$\begin{aligned}\lim_{r\to\infty} &\int_\Omega \Phi(T_{l+1}(u)) \cdot \nabla T_{l+1}(u)\Psi_l'(u)(h(u_r)\phi)\,\mathrm{d}x \\ &= \int_\Omega \Phi(T_{l+1}(u)) \cdot \nabla T_{l+1}(u)\Psi_l'(u)(h(u)\phi)\,\mathrm{d}x.\end{aligned} \tag{3.38}$$

For $m > 0$ such that $\mathrm{supp}(h) \subset [-m, m]$, T_{l+1} can be replaced by T_m in (3.38) and $\Psi_l'(u) = \Psi_l'(T_m(u)) = 0$ for $l > m$. Rewriting (3.38), we arrive at

$$\lim_{l\to\infty}\lim_{r\to\infty}\lim_{s\to\infty} L^{2,2}_{s,r,l} = 0.$$

Finally, we focus on the most important term, which is $L^1_{s,r,l}$. Let us write

$$\begin{aligned}L^1_{s,r,l} &= \int_\Omega \mathcal{A}(x, \nabla u_s) \cdot \nabla\Psi_l(u_s)h(u_r)\phi\,\mathrm{d}x \\ &\quad + \int_\Omega \mathcal{A}(x, \nabla u_s) \cdot \nabla\big(h(u_r)\phi\big)\Psi_l(u_s)\,\mathrm{d}x \\ &=: L^{1,1}_{s,r,l} + L^{1,2}_{s,r,l}.\end{aligned}$$

Convergence of the first term is quite straightforward, namely

$$\begin{aligned}&\lim_{l\to\infty}\lim_{r\to\infty}\limsup_{s\to\infty}\left|L^{1.1}_{s,r,l}\right| \\ &\quad\leqslant ||h||_{L^\infty(\Omega)}||\phi||_{L^\infty(\Omega)} \lim_{l\to\infty}\lim_{r\to\infty}\left(\sup_{s>0}\int_{\{l<|u_s|<l+1\}} \mathcal{A}_{s,l+1}(x)\cdot\nabla T_{l+1}(u_s)\,\mathrm{d}x\right) \\ &\quad\leqslant C\lim_{l\to\infty}\gamma\left(\frac{l}{m(c_1 l)}\right) \\ &\quad= 0,\end{aligned}$$

where in the last inequality, we used the energy control condition, stated in (3.4).

For $L^{1.2}_{s,r,l}$, we need to recall some facts we already know. Firstly, by (3.3) and the de la Vallée Poussin theorem (lemma 2.3) we get the uniform integrability of the sequence $\{\mathcal{A}_{s,l+1}\}_{s>0}$. But due to the weak-$*$ convergence in (3.11), the

https://doi.org/10.1017/prm.2023.113 Published online by Cambridge University Press

Dunford–Pettis theorem yields (up to a subsequence)

$$\mathcal{A}_{s,l+1} \rightharpoonup \mathcal{A}(x, \nabla T_{l+1}(u)) \text{ weakly in } L^1(\Omega;\mathbb{R}^n).$$

Furthermore, since we also know the following

$$\begin{gathered} |\Psi_l(u_s)| \leqslant 1, \\ \nabla(h(u_r)\phi) \in L^\infty(\Omega;\mathbb{R}^n), \\ \Psi_l(u_s) \xrightarrow[s\to\infty]{} \Psi_l(u) \text{ a.e. in } \Omega, \end{gathered}$$

we infer from lemma 2.8 that

$$\begin{aligned} &\lim_{r\to\infty} \limsup_{s\to\infty} \int_\Omega \mathcal{A}_{s,l+1}\Psi_l(u_s)\cdot\nabla(h(u_r)\phi)\,\mathrm{d}x \\ &\quad = \lim_{r\to\infty}\int_\Omega \mathcal{A}(x,\nabla T_{l+1}(u))\Psi_l(u)\cdot\nabla\big(h(u_r)\phi\big)\,\mathrm{d}x \\ &\quad = \int_\Omega \mathcal{A}(x,\nabla T_{l+1}(u))\Psi_l(u)\cdot\nabla\big(h(u)\phi\big)\,\mathrm{d}x. \end{aligned}$$

Choosing $l > m$, we obtain

$$\begin{aligned} &\lim_{l\to\infty}\lim_{r\to\infty}\limsup_{s\to\infty} L^{1,2}_{s,r,l} \\ &\quad = \lim_{l\to\infty}\int_\Omega \mathcal{A}(x,\nabla T_{l+1}(u))\cdot\nabla\big(h(u)\phi\big)\big)\Psi_l(u)\,\mathrm{d}x \\ &\quad = \int_\Omega \mathcal{A}(x,\nabla u)\cdot\nabla\big(h(u)\phi\big)\,\mathrm{d}x. \end{aligned}$$

Using the facts that $C_c^\infty(\Omega)\subset W_0^{1,\infty}(\Omega)$ and the gradients of functions in $V_0^1L_M(\Omega)$ can be approximated by smooth functions in the weak-$*$ topology of $L_M(\Omega;\mathbb{R}^n)$, we finally arrive at

$$\begin{aligned} &\int_\Omega \mathcal{A}(x,\nabla u)\cdot\nabla(h(u)\phi) + \Phi(u)\cdot\nabla(h(u)\phi) + b(x,u)h(u)\phi\,\mathrm{d}x \\ &\quad = \int_\Omega fh(u)\phi + F\cdot\nabla(h(u)\phi)\,\mathrm{d}x \end{aligned}$$

for every $h\in C_c^1(\mathbb{R})$ and all $\phi\in V_0^1L_M(\Omega)\cap L^\infty(\Omega)$. Thus, condition $(R2)$ is satisfied.

https://doi.org/10.1017/prm.2023.113 Published online by Cambridge University Press

Condition $(R3)$.

As in the definition of renormalized solutions, we have to show that

$$\int_{\{l<|u|<l+1\}} \mathcal{A}(x,\nabla u)\cdot\nabla u\,\mathrm{d}x = \int_{\{l<|u|<l+1\}} \mathcal{A}(x,\nabla T_{l+1}(u))\cdot\nabla T_{l+1}(u)\,\mathrm{d}x \xrightarrow{l\to\infty} 0.$$

The first thing we are going to prove is that

$$\mathcal{A}_{s,l+1}\nabla T_{l+1}(u_s) \rightharpoonup \mathcal{A}(x,\nabla T_{l+1}(u))\cdot\nabla T_{l+1}(u) \text{ weakly in } L^1(\Omega) \tag{3.39}$$

as $s\to\infty$. Here is the place where the first time we apply Young measures. We begin with showing the uniform integrability of the sequence

$$\left\{\left(\mathcal{A}_{s,l+1}-\mathcal{A}(x,\nabla T_{l+1}(u))\right)\cdot\left(\nabla T_{l+1}(u_s)-\nabla T_{l+1}(u)\right)\right\}_s.$$

At first, for every s we can make the following estimate

$$\begin{aligned}&\int_\Omega \left(\mathcal{A}_{s,l+1}-\mathcal{A}(x,\nabla T_{l+1}(u))\right)\cdot\left(\nabla T_{l+1}(u_s)-\nabla T_{l+1}(u)\right)\mathrm{d}x\\ &\quad\leqslant |J_1|+|J_2|+|J_3|+|J_4|,\end{aligned}$$

where

$$\begin{aligned}J_1 &:= \int_\Omega \mathcal{A}_{s,l+1}\cdot\nabla T_{l+1}(u_s)\,\mathrm{d}x,\\ J_2 &:= \int_\Omega \mathcal{A}_{s,l+1}\cdot\nabla T_{l+1}(u)\,\mathrm{d}x,\\ J_3 &= \int_\Omega \mathcal{A}(x,\nabla T_{l+1}(u))\cdot\nabla T_{l+1}(u_s)\,\mathrm{d}x,\\ J_4 &:= \int_\Omega \mathcal{A}(x,\nabla T_{l+1}(u))\cdot\nabla T_{l+1}(u)\,\mathrm{d}x.\end{aligned}$$

Then, using the Fenchel–Young inequality and our *a priori* estimates (3.2) and (3.3), we infer that

$$|J_1| = \left|\int_\Omega \mathcal{A}_{s,l+1}\cdot\nabla T_{l+1}(u_s)\,\mathrm{d}x\right| \leqslant 2\|\mathcal{A}(x,T_{l+1}(u_s))\|_{L_{M^*}(\Omega)}\|\nabla T_{l+1}(u_s)\|_{L_M(\Omega)} \leqslant C,$$

where C is independent of s. Notice that we may obtain the same estimate for each J_i, where $i=1,\,2,\,3,\,4$. Thus, since it does not depend on s, we get the uniform boundedness of the sequence

$$\left\{\left(\mathcal{A}_{s,l+1} - \mathcal{A}(x, \nabla T_{l+1}(u))\right) \cdot \left(\nabla T_{l+1}(u_s) - \nabla T_{l+1}(u)\right)\right\}_s$$

in $L^1(\Omega)$. Therefore, we may use Chacon's biting lemma (lemma 2.12) and lemma 2.9, up to a subsequence, get the following biting convergence

$$\begin{aligned} 0 \leqslant & \left(\mathcal{A}_{s,l+1} - \mathcal{A}(x, \nabla T_{l+1}(u))\right) \cdot \left(\nabla T_{l+1}(u_s) - \nabla T_{l+1}(u)\right) \\ \xrightarrow[s\to\infty]{b} & \int_{\mathbb{R}^n} \left(\mathcal{A}(x,\lambda) - \mathcal{A}(x, \nabla T_{l+1}(u))\right) \cdot \left(\lambda - \nabla T_{l+1}(u)\right) \mathrm{d}\nu_x(\lambda). \end{aligned} \tag{3.40}$$

Here, ν_x is a Young measure, which is generated by the sequence $\{\nabla T_{l+1}(u_s)\}$. Since $\nabla T_{l+1}(u_s) \rightharpoonup \nabla T_{l+1}(u)$ in $L^1(\Omega; \mathbb{R}^n)$ (obtained in (3.11)), the equality

$$\int_{\mathbb{R}^N} \lambda \,\mathrm{d}\nu_x(\lambda) = \nabla T_{l+1}(u)$$

holds for a.e. $x \in \Omega$. It follows that

$$\int_{\mathbb{R}^N} \mathcal{A}(x, \nabla T_{l+1}(u)) \cdot \big(\lambda - \nabla T_{l+1}(u)\big) \,\mathrm{d}\nu_x(\lambda) = 0.$$

Thus, our limit simply becomes

$$\begin{aligned} &\int_{\mathbb{R}^n} \left(\mathcal{A}(x,\lambda) - \mathcal{A}(x, \nabla T_{l+1}(u))\right) \cdot \left(\lambda - \nabla T_{l+1}(u)\right) \mathrm{d}\nu_x(\lambda) \\ &\qquad = \int_{\mathbb{R}^n} \mathcal{A}(x,\lambda) \cdot \lambda \,\mathrm{d}\nu_x(\lambda) - \int_{\mathbb{R}^N} \mathcal{A}(x,\lambda) \cdot \nabla T_{l+1}(u) \,\mathrm{d}\nu_x(\lambda). \end{aligned} \tag{3.41}$$

Now, the result in (3.4) yields uniform boundedness of the sequence $\{\mathcal{A}_{s,l+1} \cdot \nabla T_{l+1}(u_s)\}_s$ and this allows us to use Chacon's biting lemma (lemma 2.12) and lemma 2.9 to get

$$\mathcal{A}_{s,l+1} \cdot \nabla T_{l+1}(u_s) \xrightarrow[s\to\infty]{b} \int_{\mathbb{R}^N} \mathcal{A}(x,\lambda) \cdot \lambda \,\mathrm{d}\nu_x(\lambda).$$

Now we look on assumption $(A2)$. In particular, it immediately implies $\mathcal{A}_{s,l+1} \cdot \nabla T_{l+1}(u_s) \geqslant 0$. Thus, by lemma 2.10, we obtain

$$\limsup_{s\to\infty} \big(\mathcal{A}(x, \nabla T_{l+1}(u_s)) \cdot \nabla T_{l+1}(u_s)\big) \geqslant \int_{\mathbb{R}^n} \mathcal{A}(x,\lambda) \cdot \lambda \,\mathrm{d}\nu_x(\lambda). \tag{3.42}$$

Since we already considered this limit in (3.32), taking

$$\mathcal{A}_k = \mathcal{A}(x, \nabla T_{l+1}(u)) = \int_{\mathbb{R}^n} \mathcal{A}(x,\lambda) \,\mathrm{d}\nu_x(\lambda),$$

we may rewrite (3.42) as

$$\nabla T_{l+1}(u) \int_{\mathbb{R}^n} \mathcal{A}(x,\lambda)\,\mathrm{d}\nu_x(\lambda) \geqslant \int_{\mathbb{R}^n} \mathcal{A}(x,\lambda)\cdot\lambda\,\nu_x(\lambda).$$

Comparing this inequality with (3.41), we see that the limit obtained in (3.40) is less or equal to zero. Thus, we infer that

$$\Big(\mathcal{A}_{s,l+1} - \mathcal{A}(x,\nabla T_{l+1}(u))\Big)\cdot\Big(\nabla T_{l+1}(u_s) - \nabla T_{l+1}(u)\Big) \xrightarrow[s\to\infty]{b} 0.$$

Now we would like to show that there is actually a stronger convergence, namely

$$\mathcal{A}_{s,l+1}\cdot\nabla T_{l+1}(u_s) \xrightarrow[s\to\infty]{b} \mathcal{A}(x,\nabla T_{l+1}(u))\cdot\nabla T_{l+1}(u). \tag{3.43}$$

If so, then lemma 2.13 combined with (3.32) and also the weak-$*$ convergence of $\mathcal{A}$, described in (3.11), will give us (3.39).

Thus, let us prove (3.43). Observe that since $\mathcal{A}(x,\,\nabla T_{l+1}(u)) \in L_{M^*}(\Omega;\mathbb{R}^n)$, there exists a family of ascending sets $\{E_j^{l+1}\}$, such that

$$\lim_{j\to\infty} \left|E_j^{l+1}\right| = 0$$

and also

$$\mathcal{A}(x,\nabla T_{l+1}(u)) \in L^\infty(\Omega\setminus E_j^{l+1}).$$

As stated in (3.11), we have $\nabla T_{l+1}(u_s) \overset{*}{\rightharpoonup} \nabla T_{l+1}(u)$ weakly-$*$ in $L_M(\Omega;\mathbb{R}^n)$ as $s\to\infty$. Therefore, we infer that

$$\mathcal{A}(x,\nabla T_{l+1}(u))\cdot\big(\nabla T_{l+1}(u_s) - \nabla T_{l+1}(u)\big) \xrightarrow[s\to\infty]{b} 0.$$

Moreover, very similar arguments yield

$$\mathcal{A}_{s,l+1}\cdot\nabla T_{l+1}(u) \xrightarrow[s\to\infty]{b} \mathcal{A}(x,\nabla T_{l+1}(u))\cdot\nabla T_{l+1}(u).$$

Collecting the two above convergences, we arrive at (3.43).

Now is the time to make use of (3.39) and (3.4). First, observe that by the properties of truncations, for every $l\in\mathbb{N}$ we get

$$\nabla u_s = 0 \text{ a.e. in } \{x\in\Omega : |u_s| \in \{l, l+1\}\}.$$

Therefore, by (3.4) we obtain

$$\lim_{l\to\infty}\limsup_{s\to\infty}\int_{\{l-1<|u|<l+2\}} \mathcal{A}(x,\nabla u_s)\cdot\nabla u_s\,\mathrm{d}x = 0. \tag{3.44}$$

Let us define a function $G_l:\mathbb{R}\to\mathbb{R}$ by the formula:

$$G_l(r) = \begin{cases} 1 & \text{if } l\leqslant |r| \leqslant l+1 \\ 0 & \text{if } |r| < l-1 \text{ or } |r| > l+2 \\ \text{affine} & \text{otherwise} \end{cases}$$

Then we may write

$$\int_{\{l<|u|<l+1\}} \mathcal{A}(x,\nabla u)\cdot\nabla u\,\mathrm{d}x \leqslant \int_\Omega G_l(u)\mathcal{A}(x,\nabla T_{l+2}(u))\cdot\nabla T_{l+2}(u)\,\mathrm{d}x. \tag{3.45}$$

As we mentioned before, thanks to $(A2)$ we have $\mathcal{A}(x,\,\xi)\cdot\xi\geqslant 0$, hence taking a limit in (3.45), we get

$$\begin{aligned} 0 &\leqslant \lim_{l\to\infty}\int_{\{l<|u|<l+1\}} \mathcal{A}(x,\nabla u)\cdot\nabla u\,\mathrm{d}x \\ &\leqslant \lim_{l\to\infty}\int_\Omega G_l(u)\mathcal{A}(x,\nabla T_{l+2}(u))\cdot\nabla T_{l+2}(u)\,\mathrm{d}x. \end{aligned}\tag{3.46}$$

But since we have (3.39) and G_l is a bounded continuous function, we obtain

$$\begin{aligned} &\lim_{l\to\infty}\int_\Omega G_l(u)\mathcal{A}(x,\nabla T_{l+2}(u))\cdot\nabla T_{l+2}(u)\,\mathrm{d}x \\ &\quad= \lim_{l\to\infty}\lim_{s\to\infty}\int_\Omega G_l(u)\mathcal{A}(x,\nabla T_{l+2}(u_s))\cdot\nabla T_{l+2}(u_s)\,\mathrm{d}x \\ &\quad\leqslant \lim_{l\to\infty}\limsup_{s\to\infty}\int_{\{l-1<|u|<l+2\}} \mathcal{A}(x,\nabla u_s)\cdot\nabla u_s\,\mathrm{d}x \\ &\quad= 0. \end{aligned}\tag{3.47}$$

The last inequality follows directly from (3.44). By (3.47) and (3.46), we obtain

$$\lim_{l\to\infty}\int_{\{l<|u|<l+1\}} \mathcal{A}(x,\nabla u)\cdot\nabla u\,\mathrm{d}x = 0,$$

which gives condition $(R3)$. Thus, u is a renormalized solution and the proof is complete.

4. Uniqueness of renormalized solutions

Now we are ready to prove the uniqueness of renormalized solutions for problem (1.1) under the condition that $s\to b(\cdot,\,s)$ is strictly increasing. We would like to point out that our approach is much influenced by [**14**, **26**, **53**].

Proof of proposition 1.4. We define the auxiliary functions

$$H_\delta(r)=\begin{cases}0, & r<0,\\ \dfrac{r}{\delta}, & 0\leqslant r\leqslant\delta,\\ 1, & r>\delta,\end{cases}\quad h_l(r)=\begin{cases}1, & |r|\leqslant l-1,\\ l-|r|, & l-1\leqslant|r|\leqslant l,\\ 0, & r>l,\end{cases}\tag{4.1}$$

for $\delta>0$ and $l>1$. By denoting

$$Z_{l,\delta}=\{0<T_l(u_1)-T_l(u_2)<\delta\}$$

and testing equation (1.1) with $\phi_1=h_l(u_1)H_\delta(T_l(u_1)-T_l(u_2))$ and $\phi_2=h_l(u_2)H_\delta(T_l(u_1)-T_l(u_2))$ respectively, subtracting the resulting equations, we get

$$I^1_{l,\delta}+I^2_{l,\delta}+I^3_{l,\delta}+I^4_{l,\delta}+I^5_{l,\delta}=I^6_{l,\delta}+I^7_{l,\delta}+I^8_{l,\delta}\tag{4.2}$$

https://doi.org/10.1017/prm.2023.113 Published online by Cambridge University Press

with

$$
\begin{aligned}
I^1_{l,\delta} &= \int_\Omega \Big(b(x,u_1)h_l(u_1) - b(x,u_2)h_l(u_2) \Big) H_\delta\big(T_l(u_1) - T_l(u_2)\big)\,\mathrm{d}x, \\
I^2_{l,\delta} &= \int_\Omega \Big(h_l'(u_1)\mathcal{A}(x,\nabla u_1)\cdot\nabla u_1 - h_l'(u_2)\mathcal{A}(x,\nabla u_2)\cdot\nabla u_2 \Big) \\
&\quad \cdot H_\delta(T_l(u_1) - T_l(u_2))\,\mathrm{d}x, \\
I^3_{l,\delta} &= \frac{1}{\delta}\int_{Z_{l,\delta}} \Big(h_l(u_1)\mathcal{A}(x,\nabla u_1) - h_l(u_2)\mathcal{A}(x,\nabla u_2) \Big)\cdot\nabla\big(T_l(u_1) - T_l(u_2)\big)\,\mathrm{d}x, \\
I^4_{l,\delta} &= \int_\Omega \Big(h_l'(u_1)\Phi(u_1)\cdot\nabla u_1 - h_l'(u_2)\Phi(u_2)\cdot\nabla u_2 \Big)\cdot H_\delta(T_l(u_1) - T_l(u_2))\,\mathrm{d}x, \\
I^5_{l,\delta} &= \frac{1}{\delta}\int_{Z_{l,\delta}} \Big(h_l(u_1)\Phi(u_1) - h_l(u_2)\Phi(u_2) \Big)\cdot\nabla\big(T_l(u_1) - T_l(u_2)\big)\,\mathrm{d}x, \\
I^6_{l,\delta} &= \int_\Omega f(h_l(u_1) - h_l(u_2))\cdot H_\delta\big(T_l(u_1) - T_l(u_2)\big)\,\mathrm{d}x, \\
I^7_{l,\delta} &= \int_\Omega F\Big(h_l'(u_1)\cdot\nabla u_1 - h_l'(u_2)\cdot\nabla u_2 \Big)\cdot H_\delta(T_l(u_1) - T_l(u_2))\,\mathrm{d}x, \\
I^8_{l,\delta} &= \frac{1}{\delta}\int_{Z_{l,\delta}} F(h_l(u_1) - h_l(u_2))\cdot\nabla\big(T_l(u_1) - T_l(u_2)\big)\,\mathrm{d}x.
\end{aligned}
$$

Next, we are going to estimate $I^i_{l,\delta}(1 \leqslant i \leqslant 8)$ one by one.

Estimate of $I^1_{l,\delta}$. Note that we have $H_\delta(T_l(u_1) - T_l(u_2)) \to \operatorname{sign}_0^+(T_l(u_1) - T_l(u_2))$ as $\delta \to 0$. Thus, the Lebesgue-dominated convergence theorem yields

$$
\begin{aligned}
I^1_l &= \lim_{\delta\to 0} I^1_{l,\delta} \\
&= \int_\Omega \Big(b(x,u_1)h_l(u_1) - b(x,u_2)h_l(u_2) \Big) \operatorname{sign}_0^+(T_l(u_1) - T_l(u_2))\,\mathrm{d}x.
\end{aligned}
$$

Estimate of $I^2_{l,\delta}$. According to condition $(R3)$, we know that the integrand in $I^2_{l,\delta}$ is bounded in L^1 and by the same arguments as above we get

$$
\begin{aligned}
I^2_l = \lim_{\delta\to 0} I^2_{l,\delta} = \int_\Omega & \Big(h_l'(u_1)\mathcal{A}(x,\nabla u_1)\cdot\nabla u_1 - h_l'(u_2)\mathcal{A}(x,\nabla u_2)\cdot\nabla u_2 \Big) \\
&\cdot \operatorname{sign}_0^+(T_l(u_1) - T_l(u_2))\,\mathrm{d}x.
\end{aligned}
$$

Estimate of $I^3_{l,\delta}$. Observing that

$$
I^3_{l,\delta} = I^{3,1}_{l,\delta} + I^{3,2}_{l,\delta},
$$

where

$$I_{l,\delta}^{3,1} = \frac{1}{\delta}\int_{Z_{l,\delta}} (h_l(u_1) - h_l(u_2))\mathcal{A}(x, \nabla T_l(u_1))\nabla\big(T_l(u_1) - T_l(u_2)\big)\,\mathrm{d}x,$$
$$I_{l,\delta}^{3,2} = \frac{1}{\delta}\int_{Z_{l,\delta}} h_l(u_2)(\mathcal{A}(x, \nabla T_l(u_1)) - \mathcal{A}(x, \nabla T_l(u_2)))\nabla(T_l(u_1) - T_l(u_2))\,\mathrm{d}x.$$

The monotonicity of $\mathcal{A}$ implies that $I_{l,\delta}^{3,2} \geqslant 0$. Since $\|h_l'\|_\infty = 1$ due to the generalized Hölder inequality, we have

$$\begin{aligned} |I_{l,\delta}^{3,1}| &\leqslant \int_\Omega |\mathcal{A}(x, \nabla T_l(u_1))\nabla(T_l(u_1) - T_l(u_2))\chi_{Z_{l,\delta}}|\,\mathrm{d}x, \\ &\leqslant 2\|\mathcal{A}(x, \nabla T_l(u_1))\|_{L_{M^*}(\Omega)}\|\nabla(T_l(u_1) - T_l(u_2))\chi_{Z_{l,\delta}}\|_{L_M(\Omega)}. \end{aligned}$$

Notice that the constant C in (3.2) and (3.3) is independent of s, we obtain

$$\lim_{\delta\to 0} I_{l,\delta}^{3,1} = 0.$$

Estimate of $I_{l,\delta}^4$. We note that

$$\begin{aligned} I_{l,\delta}^4 &= \int_\Omega \left(h'(T_l(u_1))\Phi(u_1)\cdot\nabla u_1 - h'(T_l(u_2))\Phi(u_2)\cdot\nabla u_2\right) \\ &\qquad \cdot H_\delta(T_l(u_1) - T_l(u_2))\,\mathrm{d}x \\ &= \int_\Omega \operatorname{div}\left(\int_{T_l(u_2)}^{T_l(u_1)} h_l'(r)\Phi(r)dr\right)\cdot H_\delta(T_l(u_1) - T_l(u_2))\,\mathrm{d}x \\ &= -\int_\Omega \left(\int_{T_l(u_2)}^{T_l(u_1)} h_l'(r)\Phi(r)dr\right)\cdot\nabla H_\delta(T_l(u_1) - T_l(u_2))\,\mathrm{d}x \\ &= -\frac{1}{\delta}\int_{\{0<T_l(u_1)-T_l(u_2)<\delta\}} \left(\int_{T_l(u_2)}^{T_l(u_1)} h_l'(r)\Phi(r)dr\right)\cdot\nabla(T_l(u_1) - T_l(u_2))\,\mathrm{d}x. \end{aligned}$$

Then

$$|I_{l,\delta}^4| \leqslant \max_{s\in[-l,l]} |\Phi(s)| \int_{\{0<T_l(u_1)-T_l(u_2)<\delta\}} |\nabla(T_l(u_1) - T_l(u_2))|\,\mathrm{d}x \xrightarrow{\delta\to 0} 0,$$

since on the right-hand side we have integrals of integrable functions over shrinking sets. Therefore, $\lim_{\delta\to 0} I_{l,\delta}^4 = 0$.

Estimate of $I^5_{l,\delta}$. We can split $I^5_{l,\delta}$ as

$$I^5_{l,\delta} = I^{5,1}_{l,\delta} + I^{5,2}_{l,\delta},$$

where

$$I^{5,1}_{l,\delta} = \frac{1}{\delta}\int_{Z_{l,\delta}} h_l(u_1)\Bigg(\Phi(T_l(u_1)) - \Phi(T_l(u_2))\Bigg)\cdot\nabla\big(T_l(u_1) - T_l(u_2)\big)\,\mathrm{d}x,$$
$$I^{5,2}_{l,\delta} = \frac{1}{\delta}\int_{Z_{l,\delta}} (h_l(u_1) - h_l(u_2))\Phi(u_2)\cdot\nabla\big(T_l(u_1) - T_l(u_2)\big)\,\mathrm{d}x.$$

Since Φ is Lipschitz, we have

$$|I^{5,1}_{l,\delta}| \leqslant L_\Phi \int_{\{0<T_l(u_1)-T_l(u_2)<\delta\}} |\nabla(T_l(u_1) - T_l(u_2))|\,\mathrm{d}x$$
$$|I^{5,2}_{l,\delta}| \leqslant \max_{s\in[-l,l]}|\Phi(s)| \int_{\{0<T_l(u_1)-T_l(u_2)<\delta\}} |\nabla(T_l(u_1) - T_l(u_2))|\,\mathrm{d}x.$$

Argue as above, we have $\lim_{\delta\to 0} I^5_{l,\delta} = 0$.

Estimate of $I^6_{l,\delta}$. It can be deduced from the Lebesgue-dominated convergence theorem that

$$I^6_l = \lim_{\delta\to 0} I^6_{l,\delta} = \int_\Omega f(h_l(u_1) - h_l(u_2))\cdot \mathrm{sign}^+_0\,(T_l(u_1) - T_l(u_2))\,\mathrm{d}x.$$

Estimate of $I^7_{l,\delta}$. Similarly, we have

$$\begin{aligned} I^7_l &= \lim_{\delta\to 0} I^7_{l,\delta} \\ &= \int_\Omega F\Bigg(h'_l(u_1)\cdot\nabla u_1 - h'_l(u_2)\cdot\nabla u_2\Bigg)\cdot \mathrm{sign}^+_0\,(T_l(u_1) - T_l(u_2))\,\mathrm{d}x. \end{aligned}$$

Estimate of $I^8_{l,\delta}$. According to the fact that $\|h'_l\|_\infty = 1$ and Fenchel–Young inequality, we have

$$\begin{aligned} |I^8_{l,\delta}| &= \left|\frac{1}{\delta}\int_{Z_{l,\delta}} F(h_l(u_1) - h_l(u_2))\cdot\nabla\big(T_l(u_1) - T_l(u_2)\big)\,\mathrm{d}x\right| \\ &\leqslant \int_{Z_{l,\delta}} \Big|F\cdot\nabla(T_l(u_1) - T_l(u_2))\Big|\,\mathrm{d}x \\ &\leqslant \int_{Z_{l,\delta}} 2M^*\left(x, \frac{4}{c_1^{\mathcal{A}}}F\right) + \frac{1}{4}M(x, c_1^{\mathcal{A}}\nabla T_l(u_1)) + \frac{1}{4}M(x, c_1^{\mathcal{A}}\nabla T_l(u_2))\,\mathrm{d}x \xrightarrow{\delta\to 0} 0. \end{aligned}$$

Combining the above estimates, we find that

$$I^1_l + I^2_l + I^3_l = I^6_l + I^7_l.$$

Since $I_l^3 \geqslant 0$, we have

$$I_l^1 + I_l^2 \leqslant I_l^6 + I_l^7.$$

Thanks to condition $(R3)$, the definition of h_l and by the similar arguments as (3.25), we immediately have $I_l^2 \to 0$, $I_l^6 \to 0$, $I_l^7 \to 0$ as $l \to \infty$. Next, we focus on proving that

$$I^1 = \lim_{l\to\infty} I_l^1 = \int_\Omega (b(x,u_1) - b(x,u_2))^+ \,\mathrm{d}x.$$

For this, we notice that $\lim\limits_{l\to\infty} \operatorname{sign}_0^+ (T_l(u_1) - T_l(u_2)) = \operatorname{sign}_0^+ (u_1 - u_2)$ almost everywhere in Ω and weakly-$*$ in $L^\infty(\Omega)$. Therefore, we can pass to the limit and obtain

$$I^1 = \int_\Omega (b(x,u_1) - b(x,u_2)) \operatorname{sign}_0^+ (u_1 - u_2) \,\mathrm{d}x = \int_\Omega (b(x,u_1) - b(x,u_2))^+ \,\mathrm{d}x.$$

Thus, we have

$$\int_\Omega (b(x,u_1) - b(x,u_2))^+ \,\mathrm{d}x \leqslant 0,$$

so that $(b(x,\, u_1) - b(x,\, u_2))^+ = 0$ almost everywhere due to sign condition of b. Since b is strictly increasing with respect to the second variable, we see that $u_1 \leqslant u_2$. Considering $\phi_1 = h_l(u_1)H_\delta(T_l(u_2) - T_l(u_1))$ and $\phi_2 = h_l(u_2)H_\delta(T_l(u_2) - T_l(u_1))$ yields the opposite inequality and thus $u_1 = u_2$. $\square$.

Acknowledgements

Both of the authors thank for discussion with Professor Iwona Chlebicka. B. B. is supported by NCN Grant 2019/34/E/ST1/00120. Y. L. is supported by China Scholarship Council, No. 202106890038.

References

1 Y. Ahmida, I. Chlebicka, P. Gwiazda and A. Youssfi. Gossez's approximation theorems in Musielak–Orlicz–Sobolev spaces. *J. Funct. Anal.* **275** (2018), 2538–2571.

2 A. Alberico, I. Chlebicka, A. Cianchi and A. Zatorska-Goldstein. Fully anisotropic elliptic problems with minimally integrable data. *Calc. Var. Partial Differ. Equ.* **58** (2019), 186.

3 N. El Amarty, B. El Haji and M. E. L. Moumni. Existence of renormalized solution for nonlinear elliptic boundary value problem without Δ2-condition. *SeMA J.* **77** (2020), 389414.

4 G. Barletta and A. Cianchi. Dirichlet problems for fully anisotropic elliptic equations. *Proc. Roy. Soc. Edinburgh Sect. A* **147** (2017), 25–60.

5 P. Bénilan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre and J. Luis Vázquez. An L^1-theory of existence and uniqueness of solutions of nonlinear elliptic equations. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* **22** (1995), 241–273.

6 D. Blanchard and F. Murat. Renormalised solutions of nonlinear parabolic problems with L^1 data: existence and uniqueness. *Proc. Roy. Soc. Edinburgh Sect. A* **127** (1997), 1137–1152.

7 L. Boccardo, D. Giachetti, J. I. Diaz Dîaz and F. Murat. Existence and regularity of renormalized solutions for some elliptic problems involving derivatives of nonlinear terms. *J. Differ. Equ.* **106** (1993), 215–237.

8 V. I. Bogachev and M. A. Soares Ruas. *Measure theory.* Vol. 1 (Berlin: Springer-Verlag, 2007).

9 M. Borowski and I. Chlebicka. Modular density of smooth functions in inhomogeneous and fully anisotropic Musielak–Orlicz–Sobolev spaces. *J. Funct. Anal.* **283** (2022), 109716.

10 M. Borowski, I. Chlebicka and B. Miasojedow. Absence of Lavrentiev's gap for anisotropic functionals. arXiv:2210.15217, 2022.

11 H. Brézis. *Functional analysis Sobolev spaces and partial differential equations*. Vol. 2 (New York: Springer, 2011).

12 I. Chlebicka. A pocket guide to nonlinear differential equations in Musielak–Orlicz spaces. *Nonlinear Anal.* **175** (2018), 1–27.

13 I. Chlebicka. Measure data elliptic problems with generalized Orlicz growth. *Proc. Roy. Soc. Edinburgh Sect. A* **153** (2023), 588–618.

14 I. Chlebicka, F. Giannetti and A. Zatorska-Goldstein. A note on uniqueness for L^1-data elliptic problems with Orlicz growth. *Colloq. Math.* **168** (2022), 199–209.

15 I. Chlebicka, P. Gwiazda, A. Świerczewska-Gwiazda and A. Wróblewska-Kamińska. *Partial differential equations in anisotropic Musielak-Orlicz spaces* (Cham: Springer, 2021).

16 I. Chlebicka, P. Gwiazda and A. Zatorska-Goldstein. Well-posedness of parabolic equations in the non-reflexive and anisotropic Musielak–Orlicz spaces in the class of renormalized solutions. *J. Differ. Equ.* **265** (2018), 5716–5766.

17 I. Chlebicka, P. Gwiazda and A. Zatorska-Goldstein. Parabolic equation in time and space dependent anisotropic Musielak–Orlicz spaces in absence of Lavrentiev's phenomenon. *Ann. Inst. H. Poincaré C, Anal. Non Linéaire* **36** (2019), 1431–1465.

18 I. Chlebicka, A. Karppinen and Y. Li. A direct proof of existence of weak solutions to fully anisotropic and inhomogeneous elliptic problems. *Topol. Methods Nonlinear Anal.*

19 I. Chlebicka and P. Nayar. Essentially fully anisotropic Orlicz functions and uniqueness to measure data problem. *Math. Methods Appl. Sci.* **45** (2022), 8503–8527.

20 A. Cianchi. A fully anisotropic sobolev inequality. *Pacific J. Math.* **196** (2000), 283–295.

21 A. Cianchi. Symmetrization in anisotropic elliptic problems. *Comm. Partial Differ. Equ.* **32** (2007), 693–717.

22 M. Colombo and G. Mingione. Regularity for double phase variational problems. *Arch. Ration. Mech. Anal.* **215** (2015), 443–496.

23 D. Cruz-Uribe and A. Fiorenza. *Variable Lebesgue spaces: foundations and harmonic analysis* (Heidelberg: Springer Science & Business Media, 2013).

24 A. Denkowska, P. Gwiazda and P. Kalita. On renormalized solutions to elliptic inclusions with nonstandard growth. *Calc. Var. Partial Differ. Equ.* **60** (2021), 21.

25 A. Di Castro. Anisotropic elliptic problems with natural growth terms. *Manuscripta Math.* **135** (2011), 521–543.

26 R. Di Nardo, F. Feo and O. Guibé. Uniqueness of renormalized solutions to nonlinear parabolic problems with lower-order terms. *Proc. Roy. Soc. Edinburgh Sect. A* **143** (2013), 1185–1208.

27 L. Diening, P. Harjulehto, P. Hästö and M. Ruzicka. *Lebesgue and Sobolev spaces with variable exponents* (Heidelberg: Springer, 2011).

28 R. J. DiPerna and P.-L. Lions. On the cauchy problem for Boltzmann equations: global existence and weak stability. *Ann. of Math. (2)* **130** (1989), 321–366.

29 T. Donaldson. Nonlinear elliptic boundary value problems in Orlicz-Sobolev spaces. *J. Differ. Equ.* **10** (1971), 507–528.

30 G. Dong and X. Fang. Differential equations of divergence form in separable Musielak-Orlicz-Sobolev spaces. *Bound. Value Probl.* **2016** (2016), 1–19.

31 L. Esposito, F. Leonetti and G. Mingione. Sharp regularity for functionals with (p, q) growth. *J. Differ. Equ.* **204** (2004), 5–55.

32 X. Fan. Differential equations of divergence form in Musielak–Sobolev spaces and a sub-supersolution method. *J. Math. Anal. Appl.* **386** (2012), 593–604.

33 J.-P. Gossez. Nonlinear elliptic boundary value problems for equations with rapidly (or slowly) increasing coefficients. *Trans. Amer. Math. Soc.* **190** (1974), 163–205.

34 J.-P. Gossez. Orlicz-Sobolev spaces and nonlinear elliptic boundary value problems. *Nonlinear Anal. Funct. Spaces Appl.* (1979), 59–94.

35 P. Gwiazda, I. Skrzypczak and A. Zatorska-Goldstein. Existence of renormalized solutions to elliptic equation in Musielak–Orlicz space. *J. Differ. Equ.* **264** (2018), 341–377.

36 P. Gwiazda, P. Wittbold, A. Wróblewska and A. Zimmermann. Renormalized solutions of nonlinear elliptic problems in generalized Orlicz spaces. *J. Differ. Equ.* **253** (2012), 635–666.

37 P. Gwiazda, P Wittbold, A Wróblewska-Kamińska and A Zimmermann. Renormalized solutions to nonlinear parabolic problems in generalized Musielak–Orlicz spaces. *Nonlinear Anal.* **129** (2015), 1–36.

38 P. Harjulehto and P. Hästö. Generalized Orlicz spaces. In *Orlicz Spaces and Generalized Orlicz Spaces* (Cham: Springer, 2019), pp. 47–78.

39 P. Hästö. A fundamental condition for harmonic analysis in anisotropic generalized Orlicz spaces. *J. Geom. Anal.* **33** (2023), 7.

40 V. S. Klimov. Imbedding theorems and geometric inequalities. *Izv. Ross. Akad. Nauk Ser. Mat.* **40** (1976), 645–671.

41 Y. Li, F. Yao and S. Zhou. Entropy and renormalized solutions to the general nonlinear elliptic equations in Musielak–Orlicz spaces. *Nonlinear Anal. Real World Appl.* **61** (2021), 103330.

42 D. Liu and P. Zhao. Solutions for a quasilinear elliptic equation in Musielak–Sobolev spaces. *Nonlinear Anal. Real World Appl.* **26** (2015), 315–329.

43 P. Marcellini. Regularity and existence of solutions of elliptic equations with p, q-growth conditions. *J. Differ. Equ.* **90** (1991), 1–30.

44 G. Mingione and V. Rădulescu. Recent developments in problems with nonstandard growth and nonuniform ellipticity. *J. Math. Anal. Appl.* **501** (2021), 125197.

45 S. Müller. Variational models for microstructure and phase transitions. *Calculus of variations and geometric evolution problems* (1996), 85–210.

46 J. Musielak, *Orlicz spaces and modular spaces*. Lecture Notes in Mathematics, Vol. 1034 (Berlin: Springer-Verlag, 1983).

47 S Hadj Nassar, H Moussa and M Rhoudaf. Renormalized solution for a nonlinear parabolic problems with noncoercivity in divergence form in Orlicz spaces. *Appl. Math. Comput.* **249** (2014), 253–264.

48 P. Pedregal. *Parametrized measures and variational principles* (Basel: Birkhäuser Verlag, 1997).

49 H. Redwane. Existence results for a class of nonlinear parabolic equations in Orlicz spaces. *Electron. J. Qual. Theory Differ. Equ.* **2010** (2010), 1–19.

50 M. Skaff. Vector valued Orlicz spaces generalized N-functions. *ii. Pacific J. Math.* **28** (1969), 413–430.

51 M. Skaff. Vector valued Orlicz spaces generalized N-functions. *i. Pacific J. Math.* **28** (1969), 193–206.

52 H. Sohr. *The Navier–Stokes equations: an elementary functional analytic approach* (Basel: Springer Science & Business Media, 2012).

53 P. Wittbold and A. Zimmermann. Existence and uniqueness of renormalized solutions to nonlinear elliptic equations with variable exponents and L^1-data. *Nonlinear Anal.* **72** (2010), 2990–3008.

54 C. Zhang and X. Zhang. Renormalized solutions for the fractional $p(x)$-Laplacian equation with L^1 data. *Nonlinear Anal.* **190** (2020), 111610.

55 C. Zhang and S. Zhou. Renormalized and entropy solutions for nonlinear parabolic equations with variable exponents and L^1 data. *J. Differ. Equ.* **248** (2010), 1376–1400.

56 C. Zhang and S. Zhou. The well-posedness of renormalized solutions for a non-uniformly parabolic equation. *Proc. Amer. Math. Soc.* **145** (2017), 2577–2589.